Solving an asset pricing model with hybrid internal and external habits, and autocorrelated gaussian shocks

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April 14, 2005
Revised March 21, 2007

Abstract We derive an explicit formula for the price-dividend ratio of a generalized version of Abel's asset pricing model. This model is generalized in two ways: first, consumption (dividend) growth is assumed to be an AR(1) process subject to Gaussian random shocks, and second, the investor's preferences are allowed to be a convex combination of internal and external habits. With an internal habit weight, 50%, and a coefficient of risk aversion, 3.25, simulation results match the historic U.S. equity premium and risk free interest rate.

Journal of Economic Literature Classification Numbers: G12 · G13 · C63 · D51

Key Words: Analyticity · Asset pricing · Habits.

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1 Introduction

There are few asset pricing models that can be solved explicitly.[1] These limited models are used widely to understand applied problems in asset pricing. For example Lo, Mamayshy, Wang (2004) recently use a continuous time model with CARA utility to explain how trading cost influence asset prices and trading volume, while Easley and O'Hara (2004) use CARA utility in discrete time to discuss the role of information in the cost of capital. In this paper we derive an exact solution to a new asset pricing model in discrete time which generalizes Abel's (1990) habit persistence model. Moreover, we show that our price-dividend function is increasing and analytic for all situations that are of interest to financial economists; i.e., when consumption growth is low enough for the marginal utility of future cash flows to be positive, which includes all historic levels. Using the explicit solution of the asset pricing model, we develop a closed form formula for the distribution of returns. We characterize our closed form solution to the hybrid habit model using Mehra-Prescott's (1985) statistics for dividend growth. With an internal habit weight of 50% and a coefficient of risk aversion of 3.25, simulation results match the historic U.S. equity premium and risk free interest rate. In addition, the distribution of stock returns generated by our hybrid habit model has kurtosis which is closest to the historic monthly returns distribution. Thus, we provide a closed form solution to an asset pricing model which delivers realistic returns on stock and bonds.

Our model is based on Abel's (1990) asset pricing model, which was designed to explain the equity premium identified by Mehra and Prescott (1985).[2] In Abel's model, an investor's current investment-consumption decision depends on one of two habits: one based on the investor's consumption in the last period (internal habit) and the other based on the consumption of his neighbor in the last period (external habit).[3] The investor values a growing stream of future dividends, which provide his future consumption. In numerical simulations for a specific example, Abel matched the

[1] For example Brock (1979, 1982) solves for asset prices in a production economy with 100% depreciation and logarithmic preferences in discrete time. While Burnside (1998) is able to solve a similar model with constant relative risk aversion (CRRA). Merton (1990) and Wang (1994) solve an asset pricing model in continuous time with constant absolute utility (CARA).

[2] See Constantinides (2002), and Mehra and Prescott (2003) for a recent survey of this literature.

[3] See Constantinides (2002), and Chen and Ludvigson (2004) for recent surveys of asset pricing models based on internal or external habits. While Abel (1999) provides a closed form solution for an asset pricing model with leverage, he does not consider internal habits.

historical equity premium with a purely internal habit model. To achieve this result, he used the following in his example: the Mehra-Prescott (1985) statistics for consumption growth, a two-state Markov distribution for consumption (dividend) growth, and a relative risk aversion close to one.[4]

We generalize Abel's asset pricing model in two ways: first, consumption (dividend) growth is assumed to be an AR(1) process subject to Gaussian random shocks, and second, the investor's preferences are allowed to be a convex combination of internal and external habits.[5] Internal and external habits have different effects on an investor's consumption decision, and hence on the returns distribution. When valuing future payoffs from stocks, internal habit requires the investor to compare the *variation* in per period marginal utility across time periods, whereas external habit requires the investor to consider the *level* of per period marginal utility. The comparison of variance associated with internal habit magnifies the impact of random fluctuations in consumption (dividend) growth. This magnification has such a large impact that with 100% internal habit, it is feasible for the variation in marginal utility of the future payoff to generate infinite volatility in stock prices.[6] Our hybrid internal-external habit model produces intermediate levels of variation in stock prices. Using the Mehra and Prescott's statistics and a habit combination with 50% internal habit and a coefficient of relative risk aversion of 3.25, our hybrid internal-external habit model generates sufficient variation in marginal utility to match estimates of both the historical equity premium and risk free interest rate.

The closed form solution for our asset pricing model is characterized by a price-dividend ratio that is monotonic in consumption (dividend) growth when the marginal utility is positive.[7] The price-dividend function is also analytic in a neighborhood of consumption growth in which the marginal utility is always positive. Analyticity means that the price-dividend function can be represented

[4]He is limited to a two point distribution because he could not allow consumption growth to exceed an upper limit, thus ensuring positive marginal utility of consumption.

[5] Chen and Ludvigson (2004) compare a purely internal habit model and a purely external habit model, and find evidence that supports only the internal habit model. Korniotis (2004) allows for combinations of internal and external habits, and finds evidence that both are relevant.

[6]This explains why Abel had to limit his distribution of consumption growth and the coefficient of risk aversion in his numerical simulation. Adding the Gaussian distribution for dividend growth introduces the small chance that the *variation* in per period marginal utility is negative for annual dividend growth larger than 29% which is larger than eight standard deviations of dividend growth. We prove below that this results in an error less than 40 cents on a million dollar purchase of this stock as long as annual dividend growth is within the range $[-29\%, 29\%]$.

[7]Recall that the price-dividend ratio in Abel's model is dependent on consumption growth.

by a Taylor series in a neighborhood of every point within the interval of convergence for the price-dividend function.[8] CCCH show that under these circumstances, the price-dividend ratio is analytic for the special cases of Mehra-Prescott and external habit.[9] In these two cases, CCCH also demonstrate that the price-dividend function is analytic with a radius of convergence of infinity. With the explicit solution developed in this paper, we are able to confirm that the Taylor series of the price-dividend function, evaluated at the historic average rate of consumption growth, is identical to the explicit solution in the two cases analyzed by CCCH. In addition, we know that our explicit solution for the internal habit case applies when the marginal utility of consumption is positive. Therefore, we find that the explicit solution to our asset pricing model is an analytic function for all circumstances in which financial economists are interested.

Using the explicit solution to our hybrid asset pricing model, we construct an explicit formula for the distribution of stock returns, which is conditional on the growth of consumption (dividends). Our theoretical distribution of stock returns yields an equity premium of 4.2%, which matches the historic average of the annual U. S. stock returns from $1871 - 2002$. In addition, the expected return on a one period bond is 2.8% which also matches the historic value. Thus, the risk free rate puzzle identified by Weil (1989) in previous models, is resolved in our model. Our theoretical distribution is representative of the behavior of monthly stock returns: the kurtosis is significantly higher than its annual historical value; it is lower than the daily historical value; but it is closest to the kurtosis found in monthly U.S. stock return data from 1802 to 2003. The main drawback of our hybrid asset pricing model is that the standard deviation of bond returns is about four times too big. We suspect that the precautionary savings effect developed in Campbell and Cochrane (1999) and Cecchetti, Lam and Mark (2000) would remedy this problem. Thus, we provide a new closed form formula for the price-dividend ratio and stock returns which delivers reasonable moments for the distribution of stock returns.

For related results we refer the interested reader to the works of Abel (1990, 1999), Bansal

[8]Burnside (1998) also finds the explicit solution for the Mehra-Prescott case with Gaussian shocks. Also see Campbell (1986), Labadie (1989), Burnside (1998), Birdarkota and McCulloch (2003), and Tsionas (2003) for various versions of the Mehra-Prescott case. Gali (1994), Abel (1999), and Chan and Kogan (2002) examine the external habit case.

[9]Throughout this paper we use CCCH in referring to Calin, Chen, Cosimano, and Himonas (2004).

and Yaron (2004), Burnside (1998), Campbell and Cochrane (1999), Cecchetti, Lam and Mark (2000), Chan and Kogan (2002), Chen and Ludvigson (2004), Collard, Feve, and Ghattassi (2006), Constantinides (2002), Gali (1994), and Mehra and Prescott (2003).

In the next section we summarize our hybrid internal-external habit asset pricing model and write the Euler equation as an integral equation in a form that is easier to solve. In section 3 we state our main results concerning the integral equation for the hybrid asset pricing model. In section 4 we construct a sequence of functions which uniformly converges to a continuous solution for our asset pricing model. In section 5 we recognize a bound on our candidate solution so that we seek a solution in an appropriate vector space and use a weighted L^p space to prove uniqueness. In section 6 we examine the properties of our exact solution. In particular, we find an explicit formula for the distribution of stock returns and the equity premium. In section 7 the explicit price-dividend function and distribution of stock returns are used to determine whether our model matches the empirical distribution of stock returns using the statistics for dividend growth from Mehra and Prescott (1985). Our conclusions are summarized in the final section.

2 The Asset Pricing Model

In Abel's (1990) asset pricing model, the lifetime utility of the representative investor is given by

$$U_t \equiv \sum_{j=0}^{\infty} \beta^j \frac{[c_{t+j}/v_{t+j}]^{1-\gamma}}{1-\gamma}, \tag{2.1}$$

where $v_t = \left[c_{t-1}^{\rho} C_{t-1}^{1-\rho}\right]^{\alpha}$ captures both internal and external habits. The variable c_{t-1} is the consumer's own consumption in period $t-1$, and C_{t-1} is the average per capita consumption for the economy in period $t-1$. The former represents internal habit, while the later captures external habit. The relative risk aversion γ, the weight on the investor's combined habits α, the discount factor β, and the weight on the individual's internal habit ρ are all restricted to being greater than or equal to zero. When $\alpha = 0$, equation (1) is the Mehra-Prescott (1985) model, in which utility does not depend on habit. When $\alpha = 1$ and $\rho = 0$, the investor's consumption decision depends only on lagged aggregate consumption (external habit). Abel calls this the relative consumption case of "catching up with the Joneses." When $\alpha = 1$ and $\rho = 1$, the investor's consumption depends only on his internal habit.

We generalize Abel's (1990) model by allowing α and ρ to vary between 0 and 1, thus creating intermediate combinations of internal and external habits. Our second generalization of Abel's (1990) model is to allow dividend growth to follow an AR(1) process with Gaussian shocks.[10] Since the only source of income is the dividend from the risky security, we have $c_t = C_t = D_t$ in equilibrium. The dividend process for the risky security is

$$D_{t+1} = D_t e^{x_0 + \phi x_t + \nu_{t+1}}, \tag{2.2}$$

where x is the continuously compounded dividend growth rate, which follows an AR(1) process subject to a normally distributed random shock, ν, which has a zero mean and variance σ^2.

CCCH demonstrate that the Euler equation for equity prices satisfies the integral equation

$$P(x) = \frac{1}{\sqrt{2\pi}\sigma} \frac{K_0 e^{K_1 x}}{1 - K_2 e^{K_1 x}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}[\nu - \sigma^2(1-\gamma)]^2} \left[1 - K_2 e^{K_1(x_0 + \phi x + \nu)}\right] \left[1 + P(x_0 + \phi x + \nu)\right] d\nu \,. \tag{2.3}$$

We define $K_0 = \beta e^{x_0(1-\gamma) + \frac{\sigma^2}{2}(\gamma-1)^2}$, $K_1 = (1-\gamma)(\phi - \alpha)$, and $K_2 = \alpha\rho K_0$ to make the equation more transparent. The marginal utility for any time period under equilibrium is given by $\left[1 - K_2 e^{K_1 x}\right] c^{\alpha(\gamma-1)-\gamma}$. As a result, we refer to $1 - K_2 e^{K_1 x}\left(1 - K_2 e^{K_1(x_0+\phi x+\nu)}\right)$ as the investor's valuation (marginal utility) of the expected dividend in the current (next) period . While $K_0 e^{K_1 x}$ represents the investor's discounted (marginal) value of expected dividend growth. In both the Mehra-Prescott (1985) case and the internal habit case, $K_2 = 0$. Therefore, these factors for the variation in the current period and in the next period are both equal to 1. In this case, the investor compares only the *level* of expected marginal utility for this period and the next period to make the optimal investment (consumption) decision. The functional equations for the Mehra and Prescott case and the external habit case differ only by the value of the constant K_1. When the investor is influenced by internal habit, K_2 is positive, and the investor considers the expected *variation* in per period marginal utility in this period relative to its value in the next period.

[10] Both of these modifications were suggested in Abel's (1990) conclusion.

The marginal utility can be negative when the dividend growth rate is sufficiently high, however this is highly unlikely and does not materially effect our closed form solution. In fact, marginal utility is negative for dividend growth larger than $-\frac{\ln(K_2)}{K_1}$.[11] As a result, the Gaussian distribution yields a negative marginal utility for sufficiently high dividend growth.[12] This opens the possibility that the price-dividend ratio could be negative for sufficiently high dividend growth. However, for the parameters, that result in the historic average equity premium and risk free interest rate, the dividend growth must be greater than 0.29 for this to occur, which is more than eight standard deviations of dividend growth above its historic average, 0.017. However, the integral equation (2.3) considers all possible levels of dividend growth so that the price-dividend ratio in negative marginal utility states may have a significant effect on the price-dividend ratio in positive marginal utility states. We prove this error does not exceed 40 cents for a purchase of one million dollars of equity for dividend growth within the range of $0.017 \pm 8\sigma$. Thus, the benefit of a closed form solution with realistic equity premium and risk free interest rate significantly outweighs the minute chance that the marginal utility is negative.

The derivation of a closed-form solution for the price-dividend function is aided by defining two functions:

$$Q(x) = \frac{1 - K_2 e^{K_1 x}}{e^{K_1 x}} P(x) \tag{2.4}$$

and

$$M(x) = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}[\nu - \psi(x)]^2} \left(1 - K_2 e^{K_1 \nu}\right) d\nu = 1 - K_2 e^{K_1 \psi(x) + \frac{1}{2}\sigma^2 K_1^2}, \tag{2.5}$$

where $\psi(x) = \phi x + x_0 + \sigma^2(1-\gamma)$. $Q(x)$ represents the investor's marginal evaluation of the expected price-dividend ratio since $e^{K_1 x}$ is the investor's expected dividends and $1 - K_2 e^{K_1 x}$ captures the marginal utility of investment in the equity.[13] The expression $M(x)$ is the expected variation in next period's dividend.

[11] Abel avoided this issue, by considering only low levels of γ and a two point distribution.

[12] Samuelson (1970) first recognized the issue associated with assumed distributions for applied work and theoretical models in a static portfolio problem. See also Jin and Judd (2002) for a discussion of this issue when using the perturbation method.

[13] Strictly speaking recall that the average dividend growth rate, x_0, is included in the constant, K_0.

Using these functions, the integral equation (2.3) can be simplified to

$$Q(x) = K_0 M(x) + K_0 \frac{e^{K_1\psi(x)+\frac{1}{2}\sigma^2 K_1^2}}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}\{\nu-[\psi(x)+\sigma^2 K_1]\}^2} Q(\nu)\, d\nu\,. \tag{2.6}$$

As shown in CCCH, the benefit of this simplification is that Q, may be analyzed independently of the investor's expected marginal utility, $1 - K_2 e^{K_1 x}$. Once we know the solution for Q, we can use (2.4) to demonstrate that the price-dividend ratio is well behaved when $1 - K_2 e^{K_1 x} > 0$. Thus, we can examine all the circumstances in which Abel's asset pricing model is well defined.

3 Solving the Asset Pricing Model

In this section we solve the integral equation (2.6) in the space $CL^\infty(\mathbf{R}, e^{|y|})$ which is defined as follows.

Definition 1. *$CL^\infty(\mathbf{R}, e^{|y|})$ is the real vector space consisting of all the continuous functions f such that there are real numbers $M_f \geq 0$ and k_f with $|f(y)| \leq M_f e^{k_f|y|}$ for all $y \in \mathbf{R}$.*

The main results of our work are summarized in the following theorem.

Theorem 3.1. *If $|\phi| < 1$ and $K_4 = K_0 e^{\frac{K_1^2\sigma^2 - 2K_1(\phi-1)[x_0+\sigma^2(1-\gamma)]}{2(\phi-1)^2}} < 1$, then:*

(a) *the integral equation (2.6) has a solution given by*

$$Q(x) = K_2 + \sum_{n=0}^{\infty}\left[(K_0 - K_2)(K_0)^n\, e^{\frac{\phi^n-1}{\phi-1}K_1\psi(x)+\frac{n}{2}\sigma^2K_1^2+A_n+nB}\right], \tag{3.7}$$

where

$$A_n = \frac{K_1^2\sigma^2\phi^2}{2(\phi-1)^2(\phi^2-1)}(\phi^{2n}-1) + \frac{K_1(\phi-1)[x_0+\sigma^2(1-\gamma)] - K_1^2\sigma^2\phi}{(\phi-1)^3}(\phi^n-1) \tag{3.8}$$

and

$$B = \frac{-K_1^2\sigma^2\phi^2 + 2K_1^2\sigma^2\phi - 2K_1(\phi-1)[x_0+\sigma^2(1-\gamma)]}{2(\phi-1)^2}\,; \tag{3.9}$$

(b) *the solution (3.7) is unique in the real vector space $CL^\infty(\mathbf{R}, e^{|y|})$;*

(c) *the solution P to equation (2.3) has the following properties:*

(i) *if* $0 < K_2 \leqslant K_4$*, then* P *is positive and increasing in the interval* $(-\infty, -\frac{\ln K_2}{K_1})$ *;*

(ii) *if* $K_2 = 0$*, then* P *is positive and increasing in* $(-\infty, \infty)$ *;*

(iii) *if* $K_2 e^{K_1 x} < 1$*, then* P *is analytic in* $(-\infty, -\frac{\ln K_2}{K_1})$ *.*

Theorem 3.1 allows us to express the closed form solution to our asset pricing model by substituting the unique solution for Q in (3.7) into the equation for Q in (2.4), and solving for the price-dividend ratio P. While the expression for the price-dividend function is different from Burnside's Equation (17), it converges to the same solution in the case of no habit persistence ($\alpha = 0$). The partial sum in (7) is within seven dollars out of a billion of Burnsides equation (17) when $n = 2000$. The advantage of our solution is that we can examine the properties of the price-dividend function when the representative investor is characterized by different combinations of internal and external habits ($\alpha \neq 0$ and $\rho \neq 0$).[14]

The closed form solution for the price-dividend function allows us to investigate the properties of the price-dividend ratio, the return on stocks and bonds, and the equity premium. We show that the price-dividend function has two characteristics: it is (c-i) monotonic, (c-ii) infinitely differentiable, and (c-iii) analytic. CCCH demonstrate property (c-iii) for the Mehra-Prescott and external habit cases, without having found the exact solution. In our more general hybrid model, we find that the price-dividend ratio is positive and increasing in the domain in which the marginal utility of consumption is positive (c-i) and it is infinitely differentiable in the same domain (c-ii). This domain is also the one in which the function is analytic (c-iii). Analyticity of the price-dividend function means that we may write the price-dividend function as a Taylor series in a neighborhood of a point.[15] We show that the interval of convergence for our hybrid asset pricing model contains all historical levels of dividend growth that occurred from 1890 to 2002. Thus, the closed form solution for the price-dividend ratio for our hybrid asset pricing model exists, is unique, and satisfies properties (c-i) through (c-iii) for any circumstance that is of interest to financial economists.

Remark: A concern with our closed form solution is that it only determines the price-dividend ratio

[14]Recently, Collard, Feve and Ghattassi (2006) examine a solution of an internal habit model with a truncated normal distribution.

[15]See CCCH for a discussion of the properties of analytic functions.

in the interval $(-\infty, -\frac{\ln K_2}{K_1})$ in which the marginal utility of consumption is positive. However, the original integral equation (2.3) requires the price-dividend ratio to be evaluated at dividend growth rates in which the marginal utility is negative. To ascertain whether or not this defect is substantial we considered an alternative asset pricing problem: Let

$$P^{(1)}(x) = \frac{K_0 e^{K_1 x}}{1 - K_2 e^{K_1 x}} M(x) + \frac{1}{\sqrt{2\pi}\sigma} \frac{K_0 e^{K_1 x}}{1 - K_2 e^{K_1 x}} \int_{-\infty}^{\bar{\nu}} e^{-\frac{1}{2\sigma^2}[\nu - \psi(x)]^2} \left(1 - K_2 e^{K_1 \nu}\right) P^{(1)}(\nu)\, d\nu \,. \tag{3.10}$$

Here the dividend growth rate, $\bar{\nu}$, is chosen such that the marginal utility is always positive $K_2 e^{K_1 \bar{\nu}} < 1$. In addition, $P^{(1)}(x)$ is the solution to the integral equation (3.10) rather than (2.3). We can show that the integral equation (3.10) has a unique solution in the space $C((-\infty, \bar{\nu}])$ with $P^{(1)}(x) \leq P(x)$ for $x \in [-\bar{\nu}, \bar{\nu}]$.[16]

Using the parameters so that the equity premium and risk free interest rate matches their historic averages, we find $\bar{\nu} = 0.29$. As a result, the price-dividend function is set to zero for dividend growth which is over eight standard deviations above its historic average, 0.017. For these extreme values of dividend growth the investor freely disposes of the equity since they do not place positive marginal utility on the stock. Consequently, it is as if we set the price-dividend equal to zero for $x \in (\bar{\nu}, \infty)$.[17] Our strategy is to show that the solution $P^{(1)}(x)$ to the integral equation (3.10) is insignificantly different from our closed form solution as long as dividend growth is within the interval $x \in [-\bar{\nu}, \bar{\nu}]$. Thus, our closed form solution within the range $[-\bar{\nu}, \bar{\nu}]$ is an accurate representation of the solution to the hybrid asset pricing model that rules out negative marginal utility of consumption.

An alternative way to specify investor's behavior in the integral equation (3.10) is to change the distribution of dividend growth from a Gaussian distribution to a truncated Gaussian distribution. In this case the integral in (3.10) is divided by $f(x) = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\bar{\nu}} e^{-\frac{1}{2\sigma^2}[\nu - \psi(x)]^2}\, d\nu$ where $\bar{\nu}$ is set so that the marginal utility is always positive. For the parameters, which yield the correct equity premium and risk free interest rate, $f(x)$ is estimated to range from 0.9999999999800 to 0.9999999999999 for $x \in [-8\sigma, 8\sigma]$. In addition, the mathematical comparison between the integral equation (3.10) and

[16] We only compare $P^{(1)}$ with our closed form solution within a compact interval so that the sup norm is well defined in the calculation of the error.

[17] The $P(x)$ for $x > \bar{\nu}$ is not used in our estimation of the error since the comparison can be made using $Q(x)$ not $P(x)$. As a result, our error analysis only deals with the $P(x)$ for $x \in [-\bar{\nu}, \bar{\nu}]$.

our closed form solution increases substantially in its complexity, since the function $f(x)$ has no closed form expression. Thus, we decided to use the integral equation (3.10) to represent the true asset pricing model.

If (3.10) is viewed as the true representation of investor's behavior, then in an Appendix we provide an estimate of how close our closed form solution is to the true model for the price-dividend function within the interval $[-\bar{\nu}, \bar{\nu}]$. Below we find that the equity premium and risk free interest rate matches the historic averages over the last century in the United States for the parameters, $\gamma = 3.25$, $\beta = 0.9765$, $\rho = 0.5$, $\sigma = 0.036$, $x = 0.017$, and $\phi = -0.14$. With these parameters we find that this estimate is less than 40 cents out of a million dollars worth of stock for dividend growth within the range of $0.017 \pm 7.5\sigma$. Thus, the closed form solution in Theorem 1 is a precise representation of our hybrid asset pricing model for any situation a financial economist is concerned about.

4 Constructing a Solution to the Asset Pricing Model

The solution for Q will be the limit of a sequence of functions $\{ Q_n \mid n \in \mathbf{Z}^+ \}$ which we shall construct recursively for each $n \in \mathbf{Z}^+$, by starting with $Q_0(x) = K_2$, and setting

$$Q_{n+1}(x) = K_0 M(x) + K_0 \frac{e^{K_1\psi(x)+\frac{1}{2}\sigma^2 K_1^2}}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}\{y-[\psi(x)+\sigma^2 K_1]\}^2} Q_n(y)\, dy\,. \tag{4.11}$$

Our goal is to show that recursive formula (4.11) has a well defined limit $Q(x)$. Moreover, we would like to show that we can pass the limit under the integral sign in formula (4.11). More precisely, we would like to justify the following operations

$$\begin{aligned}
Q(x) &= \lim_{n\to\infty} Q_{n+1}(x) \\
&= \lim_{n\to\infty} \left[K_0 M(x) + K_0 \frac{e^{K_1\psi(x)+\frac{1}{2}\sigma^2 K_1^2}}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}\{y-[\psi(x)+\sigma^2 K_1]\}^2} Q_n(y)\, dy \right] \\
&= K_0 M(x) + K_0 \frac{e^{K_1\psi(x)+\frac{1}{2}\sigma^2 K_1^2}}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}\{y-[\psi(x)+\sigma^2 K_1]\}^2} \left[\lim_{n\to\infty} Q_n(y)\right] dy \\
&= K_0 M(x) + K_0 \frac{e^{K_1\psi(x)+\frac{1}{2}\sigma^2 K_1^2}}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}\{y-[\psi(x)+\sigma^2 K_1]\}^2} Q(y)\, dy\,.
\end{aligned}$$

Then, the so constracted Q will be a solution to the integral equation (2.6).

We shall begin the process of constructing the sequence (4.11) with the following lemma, which will help us find a closed form for its terms and its limit.

Lemma 4.1. *If $f_a(x) = e^{aK_1\psi(x)}$ for some $a \in \mathbf{R}$, then*

$$\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}\{y-[\psi(x)+\sigma^2K_1]\}^2} f_a(y)\,dy = C(a)e^{a\phi K_1\psi(x)}\,,$$

where $C(a) = e^{\frac{1}{2}a^2\phi^2\sigma^2K_1^2+aK_1[K_1\sigma^2\phi+\sigma^2(1-\gamma)+x_0]}$.

Proof: By the definitions of f_a and ψ, we obtain

$$\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}\{y-[\psi(x)+\sigma^2K_1]\}^2} f_a(y)\,dy = \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}\{y-[\psi(x)+\sigma^2K_1]\}^2+aK_1\psi(y)}\,dy\,.$$

and

$$\begin{aligned}&-\tfrac{1}{2\sigma^2}\{y-[\psi(x)+\sigma^2K_1]\}^2 + aK_1\psi(y)\\ &= -\tfrac{1}{2\sigma^2}\{y-[\psi(x)+\sigma^2K_1(1+a\phi)]\}^2 + a\phi K_1\psi(x)\\ &+\tfrac{1}{2}a^2\phi^2\sigma^2K_1^2 + aK_1[K_1\sigma^2\phi+\sigma^2(1-\gamma)+x_0]\,.\end{aligned}$$

Therefore, we conclude that $\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}\{y-[\psi(x)+\sigma^2K_1]\}^2} f_a(y)\,dy = C(a)e^{a\phi K_1\psi(x)}$.[18] ■

Applying Lemma 4.1 repeatedly allows us to find a closed form formula for the consecutive differences $Q_{n+1}(x) - Q_n(x)$.

Lemma 4.2. *For any $n \in \mathbf{Z}^+$, we have*

$$Q_{n+1}(x) - Q_n(x) = (K_0 - K_2)\,(K_0)^n\, e^{\frac{\phi^n-1}{\phi-1}K_1\psi(x)+\frac{n}{2}\sigma^2K_1^2+A_n+nB}\,,$$

where A_n and B are given by (3.8) and (3.9), respectively.

Proof: Note that $Q_0(x) = K_2$ and $M(x) = 1 - K_2e^{K_1\psi(x)+\frac{1}{2}\sigma^2K_1^2}$. By the recursive formula (4.11), we obtain

$$Q_1(x) = K_0M(x) + K_0\frac{e^{K_1\psi(x)+\frac{1}{2}\sigma^2K_1^2}}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}\{y-[\psi(x)+\sigma^2K_1]\}^2}K_2\,dy = K_0\,.$$

If $n = 0$, then

$$Q_1(x) - Q_0(x) = K_0 - K_2 = (K_0-K_2)\,(K_0)^0\, e^{\frac{\phi^0-1}{\phi-1}K_1\psi(x)+\frac{0}{2}\sigma^2K_1^2}\prod_{i=0}^{-1}C\left(\frac{\phi^i-1}{\phi-1}\right),$$

where $\prod_{i=0}^{-1}C\left(\frac{\phi^i-1}{\phi-1}\right) = 1$, and hence the formula in the lemma holds.

[18] By the definition of the function C, we have $C(0) = e^0 = 1$.

Suppose that for some $n \in \mathbf{Z}^+$,

$$Q_{n+1}(x) - Q_n(x) = (K_0 - K_2)(K_0)^n \, e^{\frac{\phi^n - 1}{\phi - 1} K_1 \psi(x) + \frac{n}{2}\sigma^2 K_1^2} \prod_{i=0}^{n-1} C\left(\frac{\phi^i - 1}{\phi - 1}\right).$$

We shall show that the formula for $Q_{n+2}(x) - Q_{n+1}(x)$ in the lemma also holds. By the recursive formula (4.11) and Lemma 4.1, we can find

$$\begin{aligned}
Q_{n+2}(x) - Q_{n+1}(x) &= K_0 \tfrac{e^{K_1\psi(x)+\frac{1}{2}\sigma^2K_1^2}}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}\{y-[\psi(x)+\sigma^2K_1]\}^2} [Q_{n+1}(y) - Q_n(y)]\, dy \\
&= (K_0 - K_2)(K_0)^{n+1} \, e^{\frac{n+1}{2}\sigma^2K_1^2} \textstyle\prod_{i=0}^{n-1} C\left(\frac{\phi^i-1}{\phi-1}\right) \\
&\quad \times \tfrac{e^{K_1\psi(x)}}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}\{y-[\psi(x)+\sigma^2K_1]\}^2} f_{\frac{\phi^n-1}{\phi-1}}(y)\, dy \\
&= (K_0 - K_2)(K_0)^{n+1} \, e^{\frac{n+1}{2}\sigma^2K_1^2} \textstyle\prod_{i=0}^{n-1} C\left(\frac{\phi^i-1}{\phi-1}\right) \\
&\quad \times C\left(\tfrac{\phi^n-1}{\phi-1}\right) e^{\left(\frac{\phi^n-1}{\phi-1}\phi+1\right)K_1\psi(x)} \\
&= (K_0 - K_2)(K_0)^{n+1} \, e^{\frac{\phi^{n+1}-1}{\phi-1}K_1\psi(x)+\frac{n+1}{2}\sigma^2K_1^2} \textstyle\prod_{i=0}^{n} C\left(\frac{\phi^i-1}{\phi-1}\right).
\end{aligned}$$

By the mathematical induction method, the formula

$$Q_{n+1}(x) - Q_n(x) = (K_0 - K_2)(K_0)^n \, e^{\frac{\phi^n - 1}{\phi - 1} K_1 \psi(x) + \frac{n}{2}\sigma^2 K_1^2} \prod_{i=0}^{n-1} C\left(\frac{\phi^i - 1}{\phi - 1}\right)$$

holds for all $n \in \mathbf{Z}^+$.

By the definition of the function C in Lemma 4.1, we have

$$\prod_{i=0}^{n-1} C\left(\frac{\phi^i - 1}{\phi - 1}\right) = e^{\frac{1}{2}\phi^2\sigma^2K_1^2 \sum_{i=0}^{n-1}\left(\frac{\phi^i-1}{\phi-1}\right)^2 + K_1[K_1\sigma^2\phi+\sigma^2(1-\gamma)+x_0]\sum_{i=0}^{n-1}\frac{\phi^i-1}{\phi-1}}.$$

It suffices to show that the exponent on the right hand side equals $A_n + nB$.

$$\begin{aligned}
&\tfrac{1}{2}K_1^2\sigma^2\phi^2 \textstyle\sum_{i=0}^{n-1}\left(\frac{\phi^i-1}{\phi-1}\right)^2 + K_1[K_1\sigma^2\phi + \sigma^2(1-\gamma) + x_0]\sum_{i=0}^{n-1}\frac{\phi^i-1}{\phi-1} \\
&= \tfrac{K_1^2\sigma^2\phi^2}{2(\phi-1)^2(\phi^2-1)}(\phi^{2n}-1) - \tfrac{K_1^2\sigma^2\phi^2}{(\phi-1)^3}(\phi^n-1) + \tfrac{K_1^2\sigma^2\phi^2}{2(\phi-1)^2}n \\
&\quad + \tfrac{K_1[K_1\sigma^2\phi+\sigma^2(1-\gamma)+x_0]}{(\phi-1)^2}(\phi^n-1) - \tfrac{K_1[K_1\sigma^2\phi+\sigma^2(1-\gamma)+x_0]}{\phi-1}n \\
&= \tfrac{K_1^2\sigma^2\phi^2}{2(\phi-1)^2(\phi^2-1)}(\phi^{2n}-1) + \tfrac{K_1(\phi-1)[x_0+\sigma^2(1-\gamma)]-K_1^2\sigma^2\phi}{(\phi-1)^3}(\phi^n-1) \\
&\quad + \tfrac{-K_1^2\sigma^2\phi^2+2K_1^2\sigma^2\phi-2K_1(\phi-1)[x_0+\sigma^2(1-\gamma)]}{2(\phi-1)^2}n \\
&= A_n + nB.
\end{aligned}$$

■

The explicit formula given in Lemma 4.2 allows us to determine when the series $\sum_{n=0}^{\infty}[Q_{n+1}(x) - Q_n(x)]$ is absolutely convergent.

Lemma 4.3. *If* $|\phi| < 1$ *and* $K_4 = K_0 e^{\frac{K_1^2\sigma^2 - 2K_1(\phi-1)[x_0+\sigma^2(1-\gamma)]}{2(\phi-1)^2}} < 1$, *then for any* $x \in \mathbf{R}$ *the series* $\sum_{n=0}^{\infty}[Q_{n+1}(x) - Q_n(x)]$ *is absolutely convergent.*

Proof: By Lemma 4.2, $Q_{n+1}(x) - Q_n(x) = (K_0 - K_2)(K_0)^n e^{\frac{\phi^n-1}{\phi-1}K_1\psi(x) + \frac{n}{2}\sigma^2K_1^2 + A_n + nB}$. When $|\phi| < 1$, we have $\lim_{n\to\infty}\phi^n = 0$, whence $e^{\frac{\phi^n-1}{\phi-1}K_1\psi(x)+A_n}$ is a bounded sequence. Write M for an upper bound of the sequence $e^{\frac{\phi^n-1}{\phi-1}K_1\psi(x)+A_n}$. By the definition of the constant B in Theorem 3.1, we obtain $\sigma^2K_1^2 + 2B = \frac{K_1^2\sigma^2 - 2K_1(\phi-1)[x_0+\sigma^2(1-\gamma)]}{(\phi-1)^2}$. By Lemma 4.2, for each $n \in \mathbf{Z}^+$ we have

$$\begin{aligned} |\, Q_{n+1}(x) - Q_n(x)\,| &= \left|(K_0 - K_2)(K_0)^n e^{\frac{\phi^n-1}{\phi-1}K_1\psi(x) + A_n + \frac{n}{2}(\sigma^2K_1^2+2B)}\right| \\ &\leq M\,|K_0 - K_2| \left(K_0 e^{\frac{K_1^2\sigma^2 - 2K_1(\phi-1)[x_0+\sigma^2(1-\gamma)]}{2(\phi-1)^2}}\right)^n . \end{aligned}$$

Since $K_0 e^{\frac{K_1^2\sigma^2 - 2K_1(\phi-1)[x_0+\sigma^2(1-\gamma)]}{2(\phi-1)^2}} < 1$, the series $\sum_{n=0}^{\infty}|Q_{n+1}(x) - Q_n(x)|$ is convergent by the direct comparison test for convergent series. ∎

A Closed Form Solution. We can now construct a candidate solution to the integral equation (2.6) based on the sequence Q_n in (4.11). For each $x \in \mathbf{R}$, we define

$$\begin{aligned} Q(x) &= Q_0(x) + \sum_{n=0}^{\infty}[Q_{n+1}(x) - Q_n(x)] \\ &= K_2 + \sum_{n=0}^{\infty}\left[(K_0 - K_2)(K_0)^n e^{\frac{\phi^n-1}{\phi-1}K_1\psi(x) + \frac{n}{2}\sigma^2K_1^2 + A_n + nB}\right] . \end{aligned}$$

It is evident that $Q(x) = \lim_{n\to\infty} Q_n(x)$ for any $x \in \mathbf{R}$.

Lemma 4.4. *The sequence* Q_n *uniformly converges to the continuous function* Q *on any bounded interval* $[a, b]$. *In addition, for any* $x \in \mathbf{R}$, *we have*

$$\lim_{n\to\infty} \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}\{y-[\psi(x)+\sigma^2K_1]\}^2}[Q(y) - Q_n(y)]\, dy = 0 .$$

Proof: Recall that for each $n \in \mathbf{Z}^+$, we defined

$$A_n = \frac{K_1^2\sigma^2\phi^2}{2(\phi-1)^2(\phi^2-1)}(\phi^{2n} - 1) + \frac{K_1(\phi-1)[x_0+\sigma^2(1-\gamma)] - K_1^2\sigma^2\phi}{(\phi-1)^3}(\phi^n - 1) .$$

Set $A_{\max} = \sup_{n\in\mathbf{Z}^+}|A_n|$. Since $|\phi| < 1$, we can find the following estimate for $A_{\max}$:

$$A_{\max} \leq \frac{K_1^2\sigma^2\phi^2}{2(1-\phi)^2(1-\phi^2)} + \frac{K_1(1-\phi)[x_0+\sigma^2(1-\gamma)] + K_1^2\sigma^2|\phi|}{(1-\phi)^3}(1 + |\phi|).$$

For any $N \in \mathbf{Z}^+$, we have

$$\begin{aligned} Q_N(x) &= Q_0(x) + \sum_{n=0}^{N-1}[Q_{n+1}(x) - Q_n(x)] \\ &= K_2 + \sum_{n=0}^{N-1}\left[(K_0 - K_2)\,(K_0)^n\, e^{\frac{\phi^n-1}{\phi-1}K_1\psi(x)+\frac{n}{2}\sigma^2K_1^2+A_n+nB}\right]. \end{aligned}$$

We obtain the following estimate:

$$\begin{aligned} |\,Q(x) - Q_N(x)\,| &\leq \sum_{n=N}^{\infty}\left[|K_0 - K_2|\,(K_0)^n\, e^{\frac{\phi^n-1}{\phi-1}K_1\psi(x)+A_n+\frac{n}{2}(\sigma^2K_1^2+2B)}\right] \\ &\leq |K_0 - K_2|\, e^{\frac{1+|\phi|}{1-\phi}K_1|\psi(x)|+A_{\max}} \sum_{n=N}^{\infty}\left(K_0 e^{\frac{K_1^2\sigma^2-2K_1(\phi-1)[x_0+\sigma^2(1-\gamma)]}{2(\phi-1)^2}}\right)^n \\ &= |K_0 - K_2|\, e^{\frac{1+|\phi|}{1-\phi}K_1|\psi(x)|+A_{\max}} \frac{(K_4)^N}{1-K_4}, \end{aligned} \tag{4.12}$$

where $K_4 = K_0 e^{\frac{K_1^2\sigma^2-2K_1(\phi-1)[x_0+\sigma^2(1-\gamma)]}{2(\phi-1)^2}} < 1$, according to the assumption in Theorem 3.1.

Recall that $\psi(x) = \phi x + x_0 + \sigma^2(1-\gamma)$. We have the inequality

$$|\psi(x)| \leq |\phi|\,|x| + |x_0| + \sigma^2|1-\gamma|\,. \tag{4.13}$$

Define $\psi_{\max} = \max_{x\in[a,b]}|\psi(x)|$. For any $n \in \mathbf{Z}^+$ and $x \in [a,b]$, by the estimate (4.12) we obtain

$$|\,Q(x) - Q_n(x)\,| \leq |K_0 - K_2|\, e^{\frac{1+|\phi|}{1-\phi}K_1\psi_{\max}+A_{\max}} \frac{(K_4)^n}{1-K_4}.$$

Note that $\lim_{n\to\infty}(K_4)^n = 0$. The sequence Q_n uniformly converges to Q on $[a,b]$.

Fix an $x_0 \in \mathbf{R}$. The previous argument shows that the sequence Q_n uniformly converges to Q on the bounded interval $[x_0-1, x_0+1]$. Since the Q_n are continuous, Q is continuous in the interval (x_0-1, x_0+1); in particular, Q is continuous at x_0.

We need the following Lemma in the subsequent analysis.

Lemma 4.5. *For any $A \in \mathbf{R}$, we have* $\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{-\frac{y^2}{2\sigma^2}+A|y|}\,dy \leq 2e^{\frac{\sigma^2A^2}{2}}$.

Proof: In fact,

$$\begin{aligned} &\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{-\frac{y^2}{2\sigma^2}+A|y|}\,dy \\ =\ &\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{0} e^{-\frac{y^2}{2\sigma^2}-Ay}\,dy + \frac{1}{\sqrt{2\pi}\sigma}\int_{0}^{\infty} e^{-\frac{y^2}{2\sigma^2}+Ay}\,dy \\ =\ &\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{0} e^{-\frac{1}{2\sigma^2}(y+\sigma^2A)^2+\frac{\sigma^2A^2}{2}}\,dy + \frac{1}{\sqrt{2\pi}\sigma}\int_{0}^{\infty} e^{-\frac{1}{2\sigma^2}(y-\sigma^2A)2+\frac{\sigma^2A^2}{2}}\,dy \\ \leq\ &\frac{e^{\frac{\sigma^2A^2}{2}}}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}(y+\sigma^2A)^2}\,dy + \frac{e^{\frac{\sigma^2A^2}{2}}}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}(y-\sigma^2A)^2}\,dy = 2e^{\frac{\sigma^2A^2}{2}}. \end{aligned}$$

■

Let $\epsilon > 0$. Note that $K_4 = K_0 e^{\frac{K_1^2\sigma^2 - 2K_1(\phi-1)[x_0+\sigma^2(1-\gamma)]}{2(\phi-1)^2}} < 1$. By the estimates (4.12) and (4.13), for any $n \in \mathbf{Z}^+$ and $y \in \mathbf{R}$ we have

$$\begin{aligned} |\, Q(y) - Q_n(y)\,| &\leq |K_0 - K_2|\, e^{\frac{1+|\phi|}{1-\phi}|\phi|K_1|y| + \frac{1+|\phi|}{1-\phi}K_1(|x_0|+\sigma^2|1-\gamma|)+A_{\max}}\, \frac{(K_4)^n}{1-K_4} \\ &= M_{\max}\, e^{\frac{|\phi|+\phi^2}{1-\phi}K_1|y|}(K_4)^n \leq M_{\max}\, e^{\frac{|\phi|+\phi^2}{1-\phi}K_1|y|}\,, \end{aligned}$$

where $M_{\max} = \frac{|K_0-K_2|}{1-K_4} e^{\frac{1+|\phi|}{1-\phi}K_1(|x_0|+\sigma^2|1-\gamma|)+A_{\max}}$. It follows easily from Lemma 4.5 that the improper integral $\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}\{y-[\psi(x)+\sigma^2K_1]\}^2 + \frac{|\phi|+\phi^2}{1-\phi}K_1|y|}\, dy$ converges. Thus, there is a real number $a > 0$ such that

$$\frac{M_{\max}}{\sqrt{2\pi}\sigma}\int_{-\infty}^{-a} e^{-\frac{1}{2\sigma^2}\{y-[\psi(x)+\sigma^2K_1]\}^2 - \frac{|\phi|+\phi^2}{1-\phi}K_1 y}\, dy + \frac{M_{\max}}{\sqrt{2\pi}\sigma}\int_{a}^{\infty} e^{-\frac{1}{2\sigma^2}\{y-[\psi(x)+\sigma^2K_1]\}^2 + \frac{|\phi|+\phi^2}{1-\phi}K_1 y}\, dy < \frac{\epsilon}{2}\,.$$

By Lemma 4.4, there is an integer $N > 0$ such that $|Q(y) - Q_n(y)| < \frac{\epsilon}{2}$ for any $n \geq N$ and $y \in [-a, a]$; in particular,

$$\frac{1}{\sqrt{2\pi}\sigma}\int_{-a}^{a} e^{-\frac{1}{2\sigma^2}\{y-[\psi(x)+\sigma^2K_1]\}^2}|Q(y) - Q_n(y)|\, dy \leq \frac{\epsilon}{2}\cdot\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}\{y-[\psi(x)+\sigma^2K_1]\}^2}\, dy = \frac{\epsilon}{2}\,.$$

If $n \geq N$, then

$$\begin{aligned} &\left|\tfrac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}\{y-[\psi(x)+\sigma^2K_1]\}^2}[Q(y) - Q_n(y)]\, dy\right| \\ \leq\; & \tfrac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}\{y-[\psi(x)+\sigma^2K_1]\}^2}|Q(y) - Q_n(y)|\, dy \\ \leq\; & \tfrac{M_{\max}}{\sqrt{2\pi}\sigma}\int_{-\infty}^{-a} e^{-\frac{1}{2\sigma^2}\{y-[\psi(x)+\sigma^2K_1]\}^2 - \frac{|\phi|+\phi^2}{1-\phi}K_1 y}\, dy \\ & +\tfrac{M_{\max}}{\sqrt{2\pi}\sigma}\int_{a}^{\infty} e^{-\frac{1}{2\sigma^2}\{y-[\psi(x)+\sigma^2K_1]\}^2 + \frac{|\phi|+\phi^2}{1-\phi}K_1 y}\, dy \\ & +\tfrac{1}{\sqrt{2\pi}\sigma}\int_{-a}^{a} e^{-\frac{1}{2\sigma^2}\{y-[\psi(x)+\sigma^2K_1]\}^2}|Q(y) - Q_n(y)|\, dy \\ <\; & \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} = \epsilon\,. \end{aligned}$$

Therefore, $\lim_{n\to\infty}\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}\{y-[\psi(x)+\sigma^2K_1]\}^2}[Q(y) - Q_n(y)]\, dy = 0$. ■

Before we proceed further, we consider the condition for the uniform convergence of the sequence of functions (4.11) to the marginal value of the expected price-dividend ratio, Q, and of the current price-dividend ratio, P. In (2.6), the investor compares the investment's current value $Q(x)$ with its future value $Q(\nu)$. The condition $K_4 < 1$ means that the investor places less weight on the future value. If the future value had a higher weight, the current value would tend to infinity since the future value increases with time. Consequently, a heavier weight on the future value would be comparable to making the time value of money closer to zero. This is the reason that condition $K_4 < 1$ is necessary for a finite valuation of the stock price.

We examine this condition using the parameters: $\sigma = 0.036$, $\phi = -0.14$, $x_0 = (1-\phi)0.017$. These three parameters are taken from Mehra and Prescott (1985) (MP) which represent the statistical properties of dividend growth in thier data set. For purposes of comparability we use these parameters for dividend growth throughout the paper. Figure 1 shows the three dimensional space given by the inequality $K_4 < 1$ in the variables β and γ, which allows us to visualize the condition for the existence of a finite equity price. The values of relative risk aversion, γ (gam), and the discount factor, β (beta), vary between $[0, 35]$ and $[0.94, 1]$, respectively. In general, the convergence condition is satisfied; however, as β approaches one, the condition is violated for low (≈ 1) and high (≈ 31) coefficients of relative risk aversion, γ. The investor places too much weight on the future as the discount factor approaches one.[19] In the habit persistence cases ($\alpha = 1$), the price-dividend function is convergent for the same values of β and γ, since the condition $K_4 < 1$ is not dependent on the constant K_2.

5 Uniqueness of the Closed Form Solution (3.7)

Next we shall prove that our candidate solution is a solution to the integral equation (2.6), and then we shall show that it is unique in the space $CL^\infty(\mathbf{R}, e^{|x|})$. We begin with the next lemma showing an exponential bound for the candidate solution.

Lemma 5.1. *There is a real number $M \geq 0$ such that $|Q(x)| \leq Me^{\frac{|\phi|+\phi^2}{1-\phi}K_1|x|}$ for all $x \in \mathbf{R}$.*

Proof: Recall that $M_{\max} = \frac{|K_0-K_2|}{1-K_4}e^{\frac{1+|\phi|}{1-\phi}K_1(|x_0|+\sigma^2|1-\gamma|)+A_{\max}}$ and $Q_0(x) = K_2$. The estimates (4.12) and (4.13) yield $|Q(x)| \leq |K_2| + M_{\max}e^{\frac{|\phi|+\phi^2}{1-\phi}K_1|x|}$. Set $M = |K_2| + M_{\max}$. Then $|Q(x)| \leq Me^{\frac{|\phi|+\phi^2}{1-\phi}K_1|x|}$ for any $x \in \mathbf{R}$. ∎

This bound allows us to establish that the integral in (2.6) converges. In fact, in our next lemma, we shall prove a more general result, which will be useful for our uniqueness proof.

[19] These values are slightly different from those found by Burnside for the MP case. However,we show that our condition $K_4 < 1$ is the optimal condition for the convergence of the unique solution for Q, (3.7), to the integral equation for Q, (2.6).

Lemma 5.2. *Let f be a continuous function and $p > 0$ a real number. If there are real numbers $M > 0$ and k such that $|f(y)| \leq Me^{k|y|}$ for all $y \in \mathbf{R}$, then the improper integral $\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{-\frac{y^2}{2\sigma^2}}|f(y)|^p\,dy$ converges.*

Proof: Thanks to the continuity of f, it suffices to show that $\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{-\frac{y^2}{2\sigma^2}}|f(y)|^p\,dy$ is finite. By Lemma 4.5, we have

$$\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{-\frac{y^2}{2\sigma^2}}|f(y)|^p\,dy \quad \leq \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{-\frac{y^2}{2\sigma^2}}\left(Me^{k|y|}\right)^p\,dy \leq 2M^p e^{\frac{\sigma^2p^2k^2}{2}}.$$

Therefore, the improper integral $\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{-\frac{y^2}{2\sigma^2}}|f(y)|^p\,dy$ converges. ∎

This property of our candidate solution suggests that we want to seek a solution in a vector space with weighted measure. Suppose that we consider the measurable space $(\mathbf{R}, \mathcal{B}_{\mathbf{R}}, \mu)$ with $\mathcal{B}_{\mathbf{R}}$ the Borel σ-algebra on $\mathbf{R}$. The formula

$$\mu(E) = \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{-\frac{y^2}{2\sigma^2}}\chi_E(y)\,dy \qquad \text{for any } E \in \mathcal{B}_{\mathbf{R}}$$

defines its measure, where χ_E is the characteristic function of E. It is well-known that each continuous function is (Lebesgue) measurable with respect to μ. We seek a solution to the integral equation (2.6) in the following vector space.

Definition 2. *Let $p > 1$. $CL^p(\mathbf{R}, d\mu)$ is the real vector space consisting of all the continuous functions f such that $\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{-\frac{y^2}{2\sigma^2}}|f(y)|^p\,dy$ converges, where $d\mu = e^{-\frac{y^2}{2\sigma^2}}\,dy$. We define the norm on $L^p(\mathbf{R}, d\mu)$ by*

$$\|f\|_p = \left(\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{-\frac{y^2}{2\sigma^2}}|f(y)|^p\,dy\right)^{1/p} \qquad \text{for any } f \in L^p(\mathbf{R}, d\mu).$$

By Lemmas 5.1 and 5.2, we see $Q \in CL^p(\mathbf{R}, d\mu)$ since $CL^\infty(\mathbf{R}, e^{|y|}) \subset CL^p(\mathbf{R}, d\mu)$. Our strategy is to show that the solution Q is unique in the bigger space $CL^p(\mathbf{R}, d\mu)$. Therefore, it will be unique in the smaller space $CL^\infty(\mathbf{R}, e^{|y|})$

The next step is to establish a condition under which the operator defined by the integral in (2.6) maps functions from $CL^p(\mathbf{R}, d\mu)$ to $CL^p(\mathbf{R}, d\mu)$.

Lemma 5.3. *Let $p > 1$. If $|\phi| < \sqrt{1 - \frac{1}{p}}$, then for any $f \in CL^p(\mathbf{R}, d\mu)$ the function Tf given by*

$$(Tf)(x) = K_0 \frac{e^{K_1\psi(x) + \frac{1}{2}\sigma^2 K_1^2}}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}\{y - [\psi(x) + \sigma^2 K_1]\}^2} f(y)\, dy$$

lies in $CL^p(\mathbf{R}, d\mu)$.

Proof: Set $q = \frac{p}{p-1}$ and $\Psi(x) = \psi(x) + \sigma^2 K_1$. By Hölder's inequality[20], we have the estimate

$$\begin{aligned}
&\left| \tfrac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}[y - \Psi(x)]^2} f(y)\, dy \right|^p \\
&= \left| e^{-\frac{1}{2\sigma^2}[\Psi(x)]^2} \cdot \tfrac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{-\frac{y^2}{2\sigma^2} + \frac{y}{\sigma^2}\Psi(x)} f(y)\, dy \right|^p \\
&\leq e^{-\frac{p}{2\sigma^2}[\Psi(x)]^2} \left(\tfrac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{-\frac{y^2}{2\sigma^2}} \left| e^{\frac{y}{\sigma^2}\Psi(x)} f(y) \right| dy \right)^p \\
&\leq e^{-\frac{p}{2\sigma^2}[\Psi(x)]^2} \left(\tfrac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{-\frac{y^2}{2\sigma^2}} e^{\frac{qy}{\sigma^2}\Psi(x)}\, dy \right)^{p/q} \left(\tfrac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{-\frac{y^2}{2\sigma^2}} |f(y)|^p\, dy \right) \\
&= e^{-\frac{p}{2\sigma^2}[\Psi(x)]^2} \left(\tfrac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}[y - q\Psi(x)]^2 + \frac{q^2}{2\sigma^2}[\Psi(x)]^2}\, dy \right)^{p/q} \|f\|_p^p \\
&= e^{\frac{p(q-1)}{2\sigma^2}[\Psi(x)]^2} \|f\|_p^p = e^{\frac{q}{2\sigma^2}[\Psi(x)]^2} \|f\|_p^p .
\end{aligned}$$

Note that $\psi(y) = \phi y + x_0 + \sigma^2(1-\gamma)$. Then $[\psi(y)]^2 = \phi^2 y^2 + 2\phi[x_0 + \sigma^2(1-\gamma)]y + [x_0 + \sigma^2(1-\gamma)]^2$. Consequently,

$$\begin{aligned}
&\tfrac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{-\frac{y^2}{2\sigma^2}} |(Tf)(y)|^p\, dy \\
&\leq \tfrac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{-\frac{y^2}{2\sigma^2}} (K_0)^p\, e^{pK_1\psi(y) + \frac{p}{2}\sigma^2 K_1^2} \cdot e^{\frac{q}{2\sigma^2}[\psi(y) + \sigma^2 K_1]^2} \|f\|_p^p\, dy \\
&= (K_0\|f\|_p)^p\, \tfrac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{-\frac{y^2}{2\sigma^2} + pK_1\psi(y) + \frac{p}{2}\sigma^2 K_1^2 + \frac{q}{2\sigma^2}[\psi(y) + \sigma^2 K_1]^2}\, dy.
\end{aligned}$$

Using $\frac{1}{p} + \frac{1}{q} = 1$ and $p + q = pq$, we may simplify the exponent inside the integral:

$$\begin{aligned}
&-\tfrac{y^2}{2\sigma^2} + pK_1\psi(y) + \tfrac{p}{2}\sigma^2 K_1^2 + \tfrac{q}{2\sigma^2}[\psi(y) + \sigma^2 K_1]^2 \\
&= -\tfrac{1 - q\phi^2}{2\sigma^2} \left\{ y - \tfrac{q\phi}{1 - q\phi^2}[x_0 + \sigma^2(1-\gamma) + p\sigma^2 K_1] \right\}^2 \\
&\quad + \tfrac{q}{2\sigma^2(1 - q\phi^2)}[x_0 + \sigma^2(1-\gamma) + p\sigma^2 K_1]^2 - \tfrac{p^2}{2}\sigma^2 K_1^2 .
\end{aligned}$$

Since $|\phi| < \sqrt{1 - \frac{1}{p}}$, or equivalently, $1 - q\phi^2 > 0$, we see that

$$\begin{aligned}
&\tfrac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{-\frac{y^2}{2\sigma^2}} |(Tf)(y)|^p\, dy \\
&\leq \tfrac{1}{\sqrt{1 - q\phi^2}} e^{\frac{q}{2\sigma^2(1 - q\phi^2)}[x_0 + \sigma^2(1-\gamma) + p\sigma^2 K_1]^2 - \frac{p^2}{2}\sigma^2 K_1^2} (K_0\|f\|_p)^p \\
&\leq \left(1 - \tfrac{p\phi^2}{p-1}\right)^{-\frac{1}{2}} e^{\frac{p}{2\sigma^2(p - 1 - p\phi^2)}[x_0 + \sigma^2(1-\gamma) + p\sigma^2 K_1]^2 - \frac{p^2}{2}\sigma^2 K_1^2} (K_0\|f\|_p)^p .
\end{aligned}$$

[20] See Folland (1984, p.174)

Therefore, $Tf \in CL^p(\mathbf{R}, d\mu)$. ∎

The computations in the proof of Lemma 5.3 show that $T\colon CL^p(\mathbf{R}, d\mu) \longrightarrow CL^p(\mathbf{R}, d\mu)$ is continuous linear operator and satisfies the inequality

$$\|T\|_p \leq K_0 \left(1 - \frac{p\phi^2}{p-1}\right)^{-\frac{1}{2p}} e^{\frac{1}{2\sigma^2(p-1-p\phi^2)}[x_0+\sigma^2(1-\gamma)+p\sigma^2 K_1]-\frac{p}{2}\sigma^2 K_1^2}\,. \tag{5.14}$$

This means that the linear operator is a contraction mapping when $\|T\|_p < 1$. Therefore, we have the following lemma.

Lemma 5.4. *Let $p > 1$. If we assume that $|\phi| < \sqrt{1-\frac{1}{p}}$ and*

$$K_0 \left(1 - \frac{p\phi^2}{p-1}\right)^{-\frac{1}{2p}} e^{\frac{1}{2\sigma^2(p-1-p\phi^2)}[x_0+\sigma^2(1-\gamma)+p\sigma^2 K_1]-\frac{p}{2}\sigma^2 K_1^2} < 1\,,$$

then the operator $T\colon CL^p(\mathbf{R}, d\mu) \longrightarrow CL^p(\mathbf{R}, d\mu)$ is a contraction, and therefore Q is the unique solution to the integral equation (2.6) in $CL^p(\mathbf{R}, d\mu)$.

Proof: Suppose that $P \in CL^p(\mathbf{R}, d\mu)$ is another function satisfying the integral equation (2.6). Then for any $x \in \mathbf{R}$, we have

$$\begin{aligned} P(x) - Q(x) &= K_0 \tfrac{e^{K_1\psi(x)+\frac{1}{2}\sigma^2 K_1^2}}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}\{y-[\psi(x)+\sigma^2 K_1]\}^2} [P(y) - Q(y)]\, dy \\ &= [T(P-Q)](x)\,. \end{aligned}$$

Now, using inequality (5.14), which reads $\|T\|_p < 1$, we have $\|P-Q\|_p = \|T(P-Q)\|_p < \|P-Q\|_p$, which is impossible. Therefore there is no other Q solution in $CL^p(\mathbf{R}, d\mu)$. ∎

Finally, we establish that the linear operator T can be restricted to $CL^\infty(\mathbf{R}, e^{|y|})$.

Lemma 5.5. *For any $p > 1$, we have $CL^\infty(\mathbf{R}, e^{|y|}) \subseteq CL^p(\mathbf{R}, d\mu)$ and T restricts to a linear operator on $CL^\infty(\mathbf{R}, e^{|y|})$.*

Proof: The first statement follows easily from Lemma 5.2.

Let $f \in CL^\infty(\mathbf{R}, e^{|y|})$. There are real numbers $M > 0$ and k such that $|f(y)| \leq Me^{k|y|}$ for all

$y \in \mathbf{R}$. By the definition of T in Lemma 5.3 and the result in Lemma 4.5, for any $x \in \mathbf{R}$ we have

$$\begin{aligned}
|(Tf)(x)| &\leq K_0 \frac{e^{K_1\psi(x)+\frac{1}{2}\sigma^2K_1^2}}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}\{y-[\psi(x)+\sigma^2K_1]\}^2} |f(y)|\, dy \\
&\leq MK_0 \frac{e^{K_1\psi(x)+\frac{1}{2}\sigma^2K_1^2-\frac{1}{2\sigma^2}[\psi(x)+\sigma^2K_1]^2}}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{-\frac{y^2}{2\sigma^2}+\frac{1}{\sigma^2}[\psi(x)+\sigma^2K_1]y+k|y|}\, dy \\
&\leq MK_0 \frac{e^{K_1\psi(x)+\frac{1}{2}\sigma^2K_1^2-\frac{1}{2\sigma^2}[\psi(x)+\sigma^2K_1]^2}}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{-\frac{y^2}{2\sigma^2}+\frac{1}{\sigma^2}[|\psi(x)|+\sigma^2K_1+\sigma^2k]|y|}\, dy \\
&\leq 2MK_0\, e^{K_1\psi(x)+\frac{1}{2}\sigma^2K_1^2-\frac{1}{2\sigma^2}[\psi(x)+\sigma^2K_1]^2+\frac{1}{2\sigma^2}[|\psi(x)|+\sigma^2K_1+\sigma^2k]^2} \\
&= 2MK_0\, e^{\frac{1}{2}\sigma^2(k+K_1)^2+(k+K_1)|\psi(x)|} \\
&\leq 2MK_0\, e^{\frac{1}{2}\sigma^2(k+K_1)^2+(k+K_1)|x_0+\sigma^2(1-\gamma)|}\, e^{(k+K_1)|\phi||x|}\,.
\end{aligned}$$

Therefore, $Tf \in CL^\infty(\mathbf{R}, e^{|y|})$. ∎

Thus, we can establish that our candidate solution is the unique solution to the integral equation (2.6) in the space $CL^\infty(\mathbf{R}, e^{|y|})$.

Lemma 5.6. *If $|\phi| < 1$ and $K_4 = K_0 e^{\frac{K_1^2\sigma^2-2K_1(\phi-1)[x_0+\sigma^2(1-\gamma)]}{2(\phi-1)^2}} < 1$, then the solution Q is unique in the space $CL^\infty(\mathbf{R}, e^{|y|})$.*

Proof: Suppose that $\bar{Q} \in CL^\infty(\mathbf{R}, e^{|y|})$ is another function which satisfies the integral equation (2.6). Then we have $\bar{Q} - Q = T(\bar{Q} - Q)$ (see the proof of Lemma 5.4). Note that $|\phi| < 1$. It is easy to show that

$$\lim_{p\to\infty} \left[K_0 \left(1 - \frac{p\phi^2}{p-1}\right)^{-\frac{1}{2p}} e^{\frac{1}{2\sigma^2(p-1-p\phi^2)}[x_0+\sigma^2(1-\gamma)+p\sigma^2K_1]-\frac{p}{2}\sigma^2K_1^2} \right] = 0\,.$$

Thus, we can choose a sufficiently large real number $p > 1$ such that $|\phi| < \sqrt{1-\frac{1}{p}}$ and

$$K_0 \left(1 - \frac{p}{p-1}\phi^2\right)^{-\frac{1}{2p}} e^{\frac{1}{2\sigma^2(p-1-p\phi^2)}[x_0+\sigma^2(1-\gamma)+p\sigma^2K_1]-\frac{p}{2}\sigma^2K_1^2} < 1\,.$$

By Lemma 5.5, we have $\bar{Q} - Q \in CL^p(\mathbf{R}, d\mu)$, whence Lemma 5.4 implies $\bar{Q} = Q$. ∎

Remark. If $K_4 \geqslant 1$, then the sequence $Q_n(x)$ constructed in formula (4.11) is divergent.

Note that $\lim_{n\to\infty} Q_n(x) = Q_0(x) + \sum_{n=0}^{\infty}[Q_{n+1}(x) - Q_n(x)]$ and that by Lemma 2 the n-th term of the series $\sum_{n=0}^{\infty}[Q_{n+1}(x) - Q_n(x)]$ is given by

$$(K_0 - K_2)\,(K_0)^n\, e^{\frac{\phi^n-1}{\phi-1}K_1\psi(x)+A_n+\frac{n}{2}(\sigma^2K_1^2+2B)} = (K_0 - K_2)\, e^{\frac{\phi^n-1}{\phi-1}K_1\psi(x)+A_n} (K_4)^n\,.$$

Then $Q_{n+1}(x) - Q_n(x)$ approaches $\pm\infty$ when $K_4 > 1$. As a result, $Q_{n+1}(x) - Q_n(x)$ approaches $(K_0 - K_2)\, e^{\frac{1}{1-\phi}K_1\psi(x)+\frac{K_1^2\sigma^2\phi^2}{2(\phi-1)^2(1-\phi^2)}+\frac{K_1(1-\phi)[x_0+\sigma^2(1-\gamma)]-K_1^2\sigma^2\phi}{(1-\phi)3}} \neq 0$ when $K_4 = 1$. In addition, the series $\sum_{n=0}^{\infty}[Q_{n+1}(x) - Q_n(x)]$, and hence the sequence $Q_n(x)$, is divergent.

6 Properties of Solution to the Asset Pricing Model

The exact solution for the price-dividend ratio in the hybrid asset pricing model is found by substituting the unique solution for Q in (3.7) into the equation for Q in (2.4), and solving for the price-dividend ratio P. Thus, the solution to our generalization of Abel's price-dividend function is

$$P(x) = \frac{K_2 e^{K_1 x}}{1 - K_2 e^{K_1 x}} + \frac{K_0 - K_2}{1 - K_2 e^{K_1 x}} \sum_{n=0}^{\infty} \left[(K_0)^n \, e^{\frac{\phi^{n+1}-1}{\phi-1} K_1 x + \frac{\phi^n - 1}{\phi - 1}[x_0 + \sigma^2(1-\gamma)]K_1 + \frac{n}{2}\sigma^2 K_1^2 + A_n + nB} \right]. \tag{6.15}$$

We are now able to establish properties (c-i) to (c-iii) for the price-dividend function. We start by showing that the price-dividend function is increasing in dividend growth.

Proposition 6.1. (i) *If* $0 < K_2 \leqslant K_4$*, then* P *is positive and increasing in the interval* $(-\infty, -\frac{\ln K_2}{K_1})$.

(ii) *If* $K_2 = 0$*, then* P *is positive and increasing in* $(-\infty, \infty)$.

Proof: (i) Recall that

$$P(x) = \frac{K_2 e^{K_1 x}}{1 - K_2 e^{K_1 x}} + \frac{K_0 - K_2}{1 - K_2 e^{K_1 x}} \sum_{n=0}^{\infty} \left[(K_0)^n \, e^{\frac{\phi^{n+1}-1}{\phi-1} K_1 x + \frac{\phi^n - 1}{\phi - 1}[x_0 + \sigma^2(1-\gamma)] + A_n + \frac{n}{2}(\sigma^2 K_1^2 + 2B_n)} \right].$$

Note that the functions $e^{K_1 x}$, $\frac{1}{1-K_2 e^{K_1 x}}$, and $e^{\frac{1-\phi^{n+1}}{1-\phi} K_1 x}$ for $n \in \mathbf{Z}^+$ are positive and increasing in $(-\infty, -\frac{\ln K_2}{K_1})$. Therefore, P is positive and increasing in $(-\infty, -\frac{\ln K_2}{K_1})$.

Statement (ii) can be verified in the same way. ∎

From Proposition 1 (ii), we can infer that the price-dividend function is monotonic for all values of dividend growth for the Mehra and Prescott case and the external habit version of Abel's model ($\rho = 0$ and $K_2 = 0$). However, for the internal habit version of Abel's model ($\rho = 1$), the price-dividend function is well defined only when the marginal utility of consumption is positive, i.e. when the dividend growth is in the interval $(-\infty, -\frac{\ln K_2}{K_1})$. When using the MP parameters in Abel's internal habit model, the price-dividend function is increasing only in the interval $(-\infty, 0.025)$, which means that the range of permissible dividend growth rates is less than 0.6942 standard deviations. When the dividend growth rate is 0.025, the price-dividend ratio is undefined since the the marginal utility, in the denominator in (6.15), is zero. By lowering the internal habit weight in our hybrid model to 50% ($\rho = 0.5$), the range of permissible dividend growth rates increases to a more reasonable interval $(-\infty, 0.29)$, which is more than $0.017 + 7.5\sigma$.[21]

CCCH show that the price-dividend function is analytic for both the Mehra-Prescott case and the external habit version of Abel's asset pricing model. The explicit solution allows us to confirm these results, as well as to analyze the hybrid asset pricing model. First, we demonstrate the conditions under which the price-dividend function is infinitely differentiable.

Lemma 6.2. *If $K_2 e^{K_1 x} < 1$, then P is infinitely differentiable.*

Proof: For proving this lemma it suffices to show that Q is infinitely differentiable, and

$$Q^{(k)}(x) = \delta_{0,k} K_2 + \sum_{n=0}^{\infty} \left(\frac{\phi^{n+1} - \phi}{\phi - 1} K_1 \right)^k \left[(K_0 - K_2)(K_0)^n \, e^{\frac{\phi^n - 1}{\phi - 1} K_1 \psi(x) + \frac{n}{2} \sigma^2 K_1^2 + A_n + nB} \right],$$

where $\delta_{0,k}$ is the Kronecker delta. Recall that $Q(x) = K_2 + \sum_{n=0}^{\infty} [Q_{n+1}(x) - Q_n(x)]$, where

$$Q_{n+1}(x) - Q_n(x) = (K_0 - K_2)(K_0)^n \, e^{\frac{\phi^n - 1}{\phi - 1} K_1 \psi(x) + \frac{n}{2} \sigma^2 K_1^2 + A_n + nB}.$$

It is easy to show with mathematical induction that for any integer $k \geq 1$,

$$\begin{aligned} \frac{d^k}{dx^k} [Q_{n+1}(x) - Q_n(x)] &= (K_0 - K_2)(K_0)^n \left(\frac{\phi^{n+1} - \phi}{\phi - 1} K_1 \right)^k e^{\frac{\phi^n - 1}{\phi - 1} K_1 \psi(x) + \frac{n}{2} \sigma^2 K_1^2 + A_n + nB} \\ &= \left(\frac{\phi^{n+1} - \phi}{\phi - 1} K_1 \right)^k [Q_{n+1}(x) - Q_n(x)]. \end{aligned}$$

Let $[a, b]$ be an arbitrary bounded closed interval. Set $\psi_{\max} = \max_{x \in [a,b]} |\psi(x)|$ and $A_{\max} = \sup_{n \in \mathbf{Z}^+} A_n$. For any $k \in \mathbf{Z}^+$ and $x \in [a, b]$, we have from the proof for Lemma 4.3

$$\left| \frac{d^k}{dx^k} [Q_{n+1}(x) - Q_n(x)] \right| \leq |K_0 - K_2| \, e^{\frac{2K_1}{1-\phi} \psi_{\max} + A_{\max}} \left(\frac{2K_1 |\phi|}{1 - \phi} \right)^k (K_4)^n$$

By the Weierstrass M-test, the series of functions

$$\sum_{n=0}^{\infty} \frac{d^k}{dx^k} [Q_{n+1}(x) - Q_n(x)] = \sum_{n=0}^{\infty} \left(\frac{\phi^{n+1} - \phi}{\phi - 1} K_1 \right)^k [Q_{n+1}(x) - Q_n(x)]$$

[21] In an earlier version of this paper we set $\rho = 0.71$ so that dividend growth was restricted to $(-\infty, 0.24)$ which is about seven standard deviations. In this case this estimate increases by about 10 times. We reduced ρ since the risk free interest rate was too high.

is uniformly convergent on $[a,b]$. Hence for any $k \in \mathbf{Z}^+$, we have

$$Q^{(k)}(x) = \delta_{0,k} K_2 + \sum_{n=0}^{\infty} \left(\frac{\phi^{n+1} - \phi}{\phi - 1} K_1 \right)^k \left[(K_0 - K_2)(K_0)^n e^{\frac{\phi^n - 1}{\phi - 1} K_1 \psi(x) + \frac{n}{2} \sigma^2 K_1^2 + A_n + nB} \right].$$

Thus, the function Q is infinitely differentiable. The infinite differentiability of the price-dividend function P is an immediate consequence of the infinite differentiability of Q and $\frac{e^{K_1 x}}{1 - K_2 e^{K_1 x}}$, provided that $K_2 e^{K_1 x} \neq 1$. ∎

Recall that a function f is said to be analytic at x_0 if there is a real number $r_{x_0} > 0$ such that $f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$ for any $x \in \mathbf{R}$ with $|x - x_0| < r_{x_0}$.

Proposition 6.3. *Let x_0 be a real number. If $K_2 e^{K_1 x_0} < 1$, then P is analytic at x_0.*

Proof: As before, it suffices to show that Q is analytic at any given point x_0. More precisely, we shall show that for any $x \in \mathbf{R}$

$$Q(x) = \sum_{k=0}^{\infty} \frac{Q^{(k)}(x_0)}{k!}(x - x_0)^k,$$

which reads that Q is analytic at x_0 and its power series expansion at x_0 has radius of convergence equal to ∞.

Let $R_N(x) = Q(x) - \sum_{n=0}^{N} \frac{Q^{(k)}(x_0)}{k!}(x - x_0)^k$ for any $N \in \mathbf{Z}^+$. Then there exists a real number ξ between x_0 and x such that $R_N(x) = \frac{Q^{(N+1)}(\xi)}{(N+1)!}(x - x_0)^{N+1}$. Choose a sufficiently large real number μ with $x \in [x_0 - \mu, x_0 + \mu]$. Set $\psi_{\max} = \max_{x \in [x_0 - \mu, x_0 + \mu]} |\psi(x)|$ and $A_{\max} = \sup_{n \in \mathbf{Z}^+} A_n$. For any integer $N \geq 0$, by Lemma 6.2 we have

$$\begin{aligned} |R_N(x)| &= \left| \frac{Q^{(N+1)}(\xi)}{(N+1)!}(x - x_0)^{N+1} \right| \\ &\leq \frac{1}{(N+1)!} \sum_{n=0}^{\infty} \left[\left(\frac{2K_1|\phi|}{1 - \phi} \right)^{N+1} |K_0 - K_2| (K_4)^n e^{\frac{2K_1}{1-\phi}\psi_{\max} + A_{\max}} \right] |x - x_0|^{N+1} \\ &= \frac{1}{(N+1)!} \left(\frac{2K_1|\phi(x - x_0)|}{1 - \phi} \right)^{N+1} \frac{|K_0 - K_2|}{1 - K_4} e^{\frac{2K_1}{1-\phi}\psi_{\max} + A_{\max}}, \end{aligned}$$

where $K_4 = K_0 e^{\frac{K_1^2 \sigma^2 - 2K_1(\phi - 1)[x_0 + \sigma^2(1-\gamma)]}{2(\phi-1)^2}} < 1$. Since $\lim_{N\to\infty} \frac{1}{(N+1)!} \left(\frac{2K_1|\phi(x - x_0)|}{1-\phi} \right)^{N+1} = 0$, we obtain $Q(x) = \sum_{k=0}^{\infty} \frac{Q^{(k)}(x_0)}{k!}(x - x_0)^k$. Now, the analyticity of the price-dividend function P is an immediate consequence of the analyticity of Q and $\frac{e^{K_1 x}}{1 - K_2 e^{K_1 x}}$ at x_0. ∎

Proposition 6.3 means that the price-dividend function may be represented by a Taylor series. These results for the price-dividend function are the best properties an asset pricing model may have short of an explicit solution. Analyticity of the price-dividend function allows us to represent the price-dividend function on the computer as

$$P_n^c(x) = e^{K_1 x} Q_n^c(x) = e^{K_1 x} \sum_{k=0}^{n} b_k (x - x_0)^k, \tag{6.16}$$

for $|x - x_0| < r$, where r is the radius of convergence, x_0 is a point at which the price-dividend function is analytic, and the b_k's are the solution to a linear system of equations which solves the optimal decision for the investor, (2.3). CCCH find that the series in (6.16) converges to the Taylor series for the price-dividend function with an error of less than 10^{-16} for the MP model with nine coefficients. In an earlier version of this paper we demonstrated that the taylor series approximation for the price-dividend function in CCCH was essentially identical to the closed form solution.

With the closed form solution to our asset pricing model we can obtain a closed form solution for stock returns and the equity premium using

$$R_{t+1} = \frac{D_{t+1} + P_{t+1}}{P_t} = e^{x_0 + \phi x + \nu} \frac{1 + P(x_0 + \phi x + \nu)}{P(x)}, \tag{6.17}$$

where we substitute our solution, Equation (6.15), for the current and future price-dividend ratio. The higher order moments – standard deviation, skewness and kurtosis – are found by integrating the deviation of (6.17) from its expected value, following the formulas from Mood, Graybill and Boes (1974 pp. 72-77). We calculate these moments using (6.17) where the integrals for the moments are evaluated using the evalf(Int(.)) command in Maple. This procedure allows us to control the range of integration so that the current state of dividend growth never exceeds the critical value below which the price-dividend ratio is well defined for the internal habit case, $\bar{\nu} = 0.29$.

The price of a one period zero coupon bond is equal to the expectation of the intertemporal rate of substitution, i.e., the pricing kernel.[22]

$$PB(x) = \beta \frac{e^{-\gamma x_0 + (\alpha(\gamma-1) - \gamma\phi) x_t}}{1 - K_2 e^{K_1 x}} \left[e^{\frac{1}{2}\gamma^2\sigma^2} - K_2 e^{K_1(x_0 + \phi x_t)} e^{\frac{\sigma^2}{2}(K_1 - \gamma)^2} \right]. \tag{6.18}$$

[22]See Abel (1990).

The bond returns are calculated using $R_B(x) = \frac{1}{PB(x)} - 1$. The moments for bond returns are computed using the same procedures as in the stock return.

7 Simulation

In this section we illustrate the properties of the price-dividend function for the hybrid asset pricing model using our explicit solution (6.15). We use the parameters: $\beta = 0.9765$, $\gamma = 3.25$, $\sigma = 0.036$, $\phi = -0.14$, $x_0 = (1-\phi)0.017$. The last three parameters are taken from Mehra and Prescott (1985) (MP) which represent the statistical properties of dividend growth in their data set. The first two parameters are chosen so that the equity premium and risk free interest rate matches the historic U. S. averages from 1871 to 2002 when the internal habit is 50%. We also calculate the returns on stock and bonds based on (6.17) and (6.18). As a result, we are able to identify the split between internal and external habits so that the equity premium and risk free interest rate is matched with the data.

7.1 Price-dividend function

We can see the impact of habit on the price-dividend function in Figure 2. The price-dividend function in the Mehra-Prescott case, near the bottom of the figure, is relatively flat, reflecting minimal variation in the price-dividend function, and hence variation in returns, for all reasonable levels of consumption growth. In the Mehra-Prescott case, the price-dividend ratio is 16.32 when evaluated at the historical average of consumption growth, $x = 0.017$. The historical average of the price-dividend ratio is about 22.9.[23] By adding external habit, the price-dividend function becomes steeper and its value at $x = 0.017$ increases to 41.69, which is higher than the historical average 22.9.[24] Finally, we can increase the slope of the price-dividend function further by increasing the internal habit percentage. In Figure 2 the steepest price-dividend function is for the hybrid habit model with 50% internal habit. This combination of internal and external habit was selected because

[23] We use the value based on the MP data, rather than from the Shiller data, to keep the results comparable with previous research. Using the Shiller values for mean and standard deviation of consumption growth does not significantly change the implications of our asset pricing model.

[24] We can match the historical average price-dividend level by lowering the discount factor, β, but this leads to a risk free rate that is too high.

it yields an accurate approximation of the historical equity premium and the risk free interest rate, as well as reasonable standard deviation in stock returns. In this case, the price-dividend ratio evaluated at the historical average consumption growth rate is not appreciably different from the external habit case at 41.82 (see Table 1). The standard deviation of returns is 22.9% relative to the standard deviation of dividend growth 3.6%. Further, the steeper slope indicates that the price-dividend ratio is more responsive to changes in dividend growth, and this increased response delivers a higher equity premium to compensate investors for the high standard deviation of stock returns. This increase in the slope of the price-dividend ratio reflects the lower marginal utility of consumption, $1 - K_2 e^{K_1 x} > 0$, as dividend growth increases. In particular, we see that the slope of the price-dividend function tends to infinity as dividend growth approaches 29.5%, and beyond this level the price-dividend function is not defined. We do not know the price-dividend function above this level of dividend growth since the marginal utility is negative. Yet, an error analysis in section 3 demonstrates that this does not materially effect our closed form solution since the error is less than 40 cents out of a million dollar purchase of equity for dividend growth in the interval $[-29\%, 29\%]$.

7.2 Data Sources

We now want to compare the properties of equity returns with those found in United States data. The data are from the following sources. The annual data are based on Shiller's extended historical sample for the years 1878 to 2002.[25] The monthly data for real stock returns and for the 90 day Treasury bill real interest rate series for the years $1946 - 2003$ are taken from CRSP.[26] The daily data for July, 1962 - December, 2003 are also taken from CRSP, and the 90 day Treasury bill returns come

[25]We obtained this data from Shiller's web site at http://www.econ.yale.edu/∼shiller/data.htm. In Shiller's data set, real consumption growth data are limited to the time period $1889 - 2002$.

[26]We follow Campbell, Lo and MacKinlay (1997, Chapter 1) in calculating the four moments for the returns on stocks and bonds and for the equity premium. Data is converted to real values by subtracting inflation, as measured by the monthly CPI. To make the statistics comparable, both the monthly and weekly data are annualized; the standard deviation and skewness are multiplied by the square root of the number of periods in the year, while the mean and kurtosis are multiplied by the number of periods in the year. Note that skewness and kurtosis are not annualized in the ARCH literature, so that our daily figures are larger than those usually reported. See, for example, Rosenberg and Engle (2002). We also computed the results using both continuous and discrete compounding. We did this to make sure that the results for the hybrid asset pricing model are not influenced by the use of continuous compounding, since Abel's theory uses discrete compounding.

from FRED II.[27] For daily and monthly stock market returns, we use the value weighted S&P 500 series. Results are unchanged when we use the equal weighted S&P 500 series, the equal weighted total market series, and the value weighted total market series. The monthly nominal stock return data in Table 3 come from Schwert (1989, 1990).[28]

7.3 Equity Premium and Higher Order Moments

In Table 1 we record the results for the 50% internal habit model, which we will use to explain the equity premium puzzle.[29] In this case, the standard deviations for stock returns and the equity premium are about 33% too high and the standard deviation for bond returns is about 400% too big. The skewness and kurtosis for stock returns are significantly above the historical annual averages.

The kurtosis for stock returns for the 50% internal habit case is less than 1.7% of the daily value, a relatively close 5.7 times its monthly value, and almost 227 times its annual value. The skewness has the wrong sign, and it decreases in absolute value as we go from daily to annual data.[30] The tendency for the skewness and kurtosis to move towards the normal distribution values as the frequency of observation declines is an implication of the law of large numbers.[31] The skewness and kurtosis of the equity premium are the same as for stock returns.

The results in Table 1 indicate that the hybrid asset pricing model is superior for explaining monthly stock returns. To obtain a longer time series for monthly nominal stock returns, we use Schwert (1989, 1990). In Table 3 we find that the result for this series ($1802-2003$) is roughly 70% of the theoretical kurtosis of stock returns in Table 1. However, the standard deviation is slightly

[27] See the web site http://research.stlouisfed.org/. Since the data are stated in terms of discount yields, we annualized the yield by adjusting the period from 360 to 365 days. There are 123 missing observations for the daily T-Bill rate, so that we have to delete the stock return and equity premium data for these days. This reduces the real stock return from 0.063 to 0.055. The other three moments do not change significantly.

[28] We obtained this data from Schwert's web site http://schwert.ssb.rochester.edu/mstock.htm. The data is updated after 1925 using CRSP valued weighted market index. We used the discrete compounding formula to be consistent with Schwert and with our theoretical framework.

[29] Recall that for a pure internal habit model, both α and ρ equal 1 and the price-dividend function is well defined only when consumption (dividend) growth is less than 2.50%. To expand the allowable range for consumption (dividend) growth, we reduce the weight on the internal habit ρ to 50%. This weight is used because with this combination the hybrid model matches the historical equity premium of 4.2% and the risk free interest rate 2.8%.

[30] This may be a by-product of the negative autocorrelation of dividend growth found in the Mehra and Prescott data.

[31] Bollerslev et al (1994, p. 3000) show that the kurtosis of a GARCH(1,1) process converges to its value for a normal distribution, 3, as the frequency of observation decreases.

higher over the monthly sample in Table 1. We also see that the skewness is positive for this sample. Finally, the kurtosis is significantly higher for the longer sample. Thus, the hybrid asset pricing model appears to best fit the longer horizon monthly stock return series ($1802-2003$).

While we are able to match the risk free interest rate, we find that the standard deviation of the return on bonds is still too high. This problem of excessive standard deviation of bond returns in habitual preference models has been found in past research. Previous researchers have introduced precautionary savings in order to lower the mean and standard deviation of bond returns, while still matching the equity premium. Campbell and Cochrane (1999) use preferences that place a higher (lower) weight on the smaller (larger) deviations of consumption (dividend) growth from habitual levels.[32] Cecchetti, Lam and Mark (2000) use distorted beliefs, with excessive pessimism (optimism) for expansion (contraction) in the Mehra-Prescott case. We plan to introduce precautionary savings into the hybrid asset pricing model in future research.

8 Conclusion

We derive an explicit formula for the solution to the price-dividend ratio of a generalized Abel asset pricing model. Abel's model is generalized in two ways: first, consumption (dividend) growth is assumed to be an AR(1) process subject to Gaussian random shocks, and second, the investor's preferences are allowed to be a convex combination of internal and external habits. We show that our price-dividend function is increasing, and analytic for all situations that are of interest to financial economists; i.e., when consumption growth is low enough for the marginal utility of future cash flows to be positive, which includes all historic levels. The Gaussian distribution assigns positive probability to dividend growth, which is high enough to make the marginal utility of investment negative when dividend growth is greater than eight standard deviations. However, we prove that the resulting error in our closed form solution is less than 40 cents out of a million dollars as long as dividend growth is inside $\pm$ eight standard deviations. From the closed form solution of the asset pricing model, we develop a closed form solution for the distribution of returns. We characterize

[32] In the original Campbell and Cochrane model the standard deviation of bond returns was zero. Wachter (2005) allows for more realistic fluctuations in bond returns.

our closed form solution to the hybrid habit model using Mehra-Prescott's (1985) data set. With an internal habit weight of 50%, and a coefficient of risk aversion 3.25 simulation results match the historic U.S. equity premium and risk free interest rate. In addition, the distribution of stock returns generated by our hybrid habit model has kurtosis which is closest to the level of the historic monthly returns distribution.

Although the magnitude of our theoretical equity premium is correct, the higher moments of the distributions of one period bond returns are too high. Previous studies find that a precautionary savings effect reduces excessive moments in bond returns, which suggests that this effect may offer a solution to the problem in our model. Both Campbell and Cochrane (1999) and Cecchetti, Lam and Mark (2000) develop models that incorporate precautionary savings. Their models put more (less) weight on low (high) random shocks to consumption growth. This property generates precautionary savings, which would control the standard deviation of bond returns. It is possible that precautionary savings models may also offer a way to control the excessive kurtosis of bond returns. In future research we plan to integrate the precautionary savings of Campbell and Cochrane into our hybrid asset pricing model so as to reduce the higher moments of bond returns.

Acknowledgments We would like to thank session participants at the January 2006 meetings of the Econometric Society for helpful comments. Discussions with Michael Gapen, Thomas Gresik, George Korniotos, Yogo Motohiro, Jun Pan, and the editor, Charalambos D. Aliprantis helped in the preparation of this work. Any remaining errors are our responsibility. Tom Cosimano received financial support from the Center for Research in Financial Services Industry at The University of Notre Dame.

Error Estimation for The Hybrid Habit Asset Pricing Model

In this appendix with prove the following proposition for $P^{(1)}$.

Proposition 8.1. *For $-\bar{\nu} \leqslant x \leqslant \bar{\nu}$,*

$$\|P^{(1)}(x) - P(x)\| \leqslant \frac{e^{K_1\bar{\nu}}}{1 - K_2 e^{K_1\bar{\nu}}} \frac{e^{\frac{1+|\phi|}{1-\phi}K_1[|\phi|\bar{\nu}+x_0+\sigma^2(1-\gamma)]+A_{\max}}}{1 - K_0 e^{\frac{K_1^2\sigma^2 - 2K_1(\phi-1)[x_0+\sigma^2(1-\gamma)]}{2(\phi-1)^2}}} \cdot \|E\| \tag{8.19}$$

where $E(x) = \frac{1}{\sqrt{2\pi}\sigma}\left\{\int_{-\infty}^{-\bar{\nu}} e^{-\frac{1}{2\sigma^2}[\nu-\psi(x)-K_1\sigma^2]^2} Q(\nu)\,d\nu + \int_{\bar{\nu}}^{\infty} e^{-\frac{1}{2\sigma^2}[\nu-\psi(x)-K_1\sigma^2]^2} Q(\nu)\,d\nu,\right\}$ *and* $A_{\max}$ *is defined in Lemma 4.*[33]

Proof:

1. Assumptions in Abel's model:

   (a) $|\phi| < 1$

   (b) $K_4 := K_0 e^{\frac{K_1^2\sigma^2 - 2K_1(\phi-1)[x_0+\sigma^2(1-\gamma)]}{2(\phi-1)^2}} < 1.$

   (c) $0 < K_2 \leqslant K_0 e^{\frac{1}{2}\sigma^2(1-\gamma)^2}$.

2. The integral equation (3) in Abel's model is given by:

$$P(x) = \frac{1}{\sqrt{2\pi}\sigma} \frac{K_0 e^{K_1 x}}{1 - K_2 e^{K_1 x}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}[\nu-\sigma^2(1-\gamma)]^2} \left[1 - K_2 e^{K_1(x_0+\phi x+\nu)}\right] [1 + P(x_0 + \phi x + \nu)]\,d\nu.$$

3. Change of variable in the integral, i.e., $\mu = x_0 + \phi x + \nu$:

$$\begin{aligned}
P(x) &= \frac{1}{\sqrt{2\pi}\sigma} \frac{K_0 e^{K_1 x}}{1 - K_2 e^{K_1 x}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}[\nu-\phi x-x_0-\sigma^2(1-\gamma)]^2} \left(1 - K_2 e^{K_1\nu}\right) [1 + P(\nu)]\,d\nu \\
&= \frac{1}{\sqrt{2\pi}\sigma} \frac{K_0 e^{K_1 x}}{1 - K_2 e^{K_1 x}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}[\nu-\psi(x)]^2} \left(1 - K_2 e^{K_1\nu}\right) [1 + P(\nu)]\,d\nu \\
&= \frac{K_0 e^{K_1 x}}{1 - K_2 e^{K_1 x}} M(x) + \frac{1}{\sqrt{2\pi}\sigma} \frac{K_0 e^{K_1 x}}{1 - K_2 e^{K_1 x}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}[\nu-\psi(x)]^2} \left(1 - K_2 e^{K_1\nu}\right) P(\nu)\,d\nu\,,
\end{aligned}$$

[33] For any $f \in C([-\bar{\nu}, \bar{\nu}])$, we define $\|f\| := \max_{-\bar{\nu}\leq x\leq\bar{\nu}} |f(x)|$.

where $\psi(x) = \phi x + x_0 + \sigma^2(1-\gamma)$ and

$$
\begin{aligned}
M(x) &= \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}[\nu-\psi(x)]^2}(1-K_2e^{K_1\nu})\,d\nu \\
&= 1 - \frac{K_2}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}[\nu-\psi(x)]^2+K_1\nu}\,d\nu \\
&= 1 - \frac{K_2}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}[\nu-\psi(x)-K_1\sigma^2]^2+K_1\psi(x)+\frac{1}{2}K_1^2\sigma^2}\,d\nu \\
&= 1 - K_2e^{K_1\psi(x)+\frac{1}{2}K_1^2\sigma^2}\,.
\end{aligned}
$$

4. Choose a $\bar{\nu} > 0$ such that $\max\{\, K_2e^{K_1\bar{\nu}}, K_2e^{K_1[|\phi|\bar{\nu}+x_0+\sigma^2(1-\gamma)]+\frac{1}{2}K_1^2\sigma^2}\,\} < 1$. If $P(x)$ denotes the solution of Equation (3), then $M(x) \geqslant 0$ and $P(x) \geqslant 0$ for $-\bar{\nu} \leqslant x \leqslant \bar{\nu}$.

5. Rather than directly compare the solutions to the integral equations (10) with (3) we consider a third intergral equation: For $-\bar{\nu} \leqslant x \leqslant \bar{\nu}$,

$$
P^{(2)}(x) = \frac{K_0e^{K_1x}}{1-K_2e^{K_1x}}M(x) + \frac{1}{\sqrt{2\pi}\sigma}\frac{K_0e^{K_1x}}{1-K_2e^{K_1x}}\int_{-\bar{\nu}}^{\bar{\nu}} e^{-\frac{1}{2\sigma^2}[\nu-\psi(x)]^2}\left(1-K_2e^{K_1\nu}\right)P^{(2)}(\nu)\,d\nu\,.
$$

   At the end of completion of this proof we then establish the result for $P^{(1)}$.

6. Let $Q(x) := \frac{1-K_2e^{K_1x}}{e^{K_1x}}P(x)$.

7. The equation in step (3) is equivalent to:

$$
\begin{aligned}
Q(x) &= K_0M(x) + \frac{K_0}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}[\nu-\psi(x)]^2+K_1\nu}Q(\nu)\,d\nu \\
&= K_0M(x) + \frac{K_0e^{K_1\psi(x)+\frac{1}{2}K_1^2\sigma^2}}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}[\nu-\psi(x)-K_1\sigma^2]^2}Q(\nu)\,d\nu\,.
\end{aligned}
$$

8. Equation in step (5) is equivalent to:

$$
Q^{(2)}(x) = K_0M(x) + \frac{K_0e^{K_1\psi(x)+\frac{1}{2}K_1^2\sigma^2}}{\sqrt{2\pi}\sigma}\int_{-\bar{\nu}}^{\bar{\nu}} e^{-\frac{1}{2\sigma^2}[\nu-\psi(x)-K_1\sigma^2]^2}Q^{(2)}(\nu)\,d\nu\,.
$$

9. Let $Q_0(x) = K_2$ for all $x \in \mathbf{R}$ and

$$
Q_{n+1}(x) = K_0M(x) + \frac{K_0e^{K_1\psi(x)+\frac{1}{2}K_1^2\sigma^2}}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}[\nu-\psi(x)-K_1\sigma^2]^2}Q_n(\nu)\,d\nu \ \text{ for } n \in \mathbf{Z}^+\,.
$$

10. Let $Q_0^{(2)}(x) = K_2$ for all $-\bar{\nu} \leqslant x \leqslant \bar{\nu}$ and

$$Q_{n+1}^{(2)}(x) = K_0 M(x) + \frac{K_0 e^{K_1\psi(x)+\frac{1}{2}K_1^2\sigma^2}}{\sqrt{2\pi}\sigma} \int_{-\bar{\nu}}^{\bar{\nu}} e^{-\frac{1}{2\sigma^2}[\nu-\psi(x)-K_1\sigma^2]^2} Q_n^{(2)}(\nu)\, d\nu \ \text{ for } n \in \mathbf{Z}^+ .$$

11. The sequence $\{ Q_n(x) - Q_n^{(2)}(x) \}_{n=0}^{\infty}$ satisfies the equation:

$$\begin{aligned} Q_{n+1}(x) - Q_{n+1}^{(2)}(x) = {} & \frac{K_0 e^{K_1\psi(x)+\frac{1}{2}K_1^2\sigma^2}}{\sqrt{2\pi}\sigma} \int_{-\bar{\nu}}^{\bar{\nu}} e^{-\frac{1}{2\sigma^2}[\nu-\psi(x)-K_1\sigma^2]^2} [Q_n(\nu) - Q_n^{(2)}(\nu)]\, d\nu \\ & + \frac{K_0 e^{K_1\psi(x)+\frac{1}{2}K_1^2\sigma^2}}{\sqrt{2\pi}\sigma} \int_{-\infty}^{-\bar{\nu}} e^{-\frac{1}{2\sigma^2}[\nu-\psi(x)-K_1\sigma^2]^2} Q_n(\nu)\, d\nu \\ & + \frac{K_0 e^{K_1\psi(x)+\frac{1}{2}K_1^2\sigma^2}}{\sqrt{2\pi}\sigma} \int_{\bar{\nu}}^{\infty} e^{-\frac{1}{2\sigma^2}[\nu-\psi(x)-K_1\sigma^2]^2} Q_n(\nu)\, d\nu \end{aligned}$$

for $n \in \mathbf{Z}^+$.

12. (a) Under the assumption (c) in part 1, $0 \leqslant Q_n(x) \leqslant Q_{n+1}(x)$ for $n \in \mathbf{Z}^+$.

(b) Under the assumption in part 4, $0 \leqslant Q_n^{(2)}(x) \leqslant Q_{n+1}^{(2)}(x)$ for $-\bar{\nu} \leqslant x \leqslant \bar{\nu}$ and $n \in \mathbf{Z}^+$.

(c) For $-\bar{\nu} \leqslant x \leqslant \bar{\nu}$ and $n \in \mathbf{Z}^+$, we have $Q_n(x) - Q_n^{(2)}(x) \geqslant 0$.

(d) By the argument in sections (4) and (5) of CCH the sequence $\{ Q_n(x) \}_{n=0}^{\infty}$ absolutely converges to the solution of the equation in part 5, say $Q(x)$.

(e) For $-\bar{\nu} \leqslant x \leqslant \bar{\nu}$ and $n \in \mathbf{Z}^+$, we have $0 \leqslant Q_n^{(2)}(x) \leqslant Q_n(x) \leqslant Q(x)$.

(f) The sequence $\{ Q_n^{(2)}(x) \}_{n=0}^{\infty}$ absolutely and uniformly converges to a continuous solution of Equation 5, say $Q^{(2)}(x)$, for $-\bar{\nu} \leqslant x \leqslant \bar{\nu}$ (see part 15: the proof of uniqueness) for a similar argument.

13. Recall from Lemma 1 that if $f_a(x) = e^{aK_1\psi(x)}$ for some $a \in \mathbf{R}$, then

$$\frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}[\nu-\psi(x)-\sigma^2 K_1]^2} f_a(\nu)\, d\nu = C(a) e^{a\phi K_1 \psi(x)} ,$$

where from the proofs of Lemma 2 and 4 $C(a) = e^{\frac{1}{2}a^2\phi^2\sigma^2K_1^2 + aK_1[K_1\sigma^2\phi + \sigma^2(1-\gamma)+x_0]}$. For each $n \in \mathbf{Z}^+$, we have

$$\prod_{i=0}^{n-1} C\left(\frac{\phi^i - 1}{\phi - 1}\right) = e^{A_n + nB} ,$$

$$A_n = \frac{K_1^2\sigma^2\phi^2}{2(\phi-1)^2(\phi^2-1)}(\phi^{2n}-1) + \frac{K_1(\phi-1)[x_0+\sigma^2(1-\gamma)] - K_1^2\sigma^2\phi}{(\phi-1)^3}(\phi^n-1)\,,$$

$$B = \frac{-K_1^2\sigma^2\phi^2 + 2K_1^2\sigma^2\phi - 2K_1(\phi-1)[x_0+\sigma^2(1-\gamma)]}{2(\phi-1)^2}\,.$$

$$e^{\frac{\phi^k-1}{\phi-1}K_1\psi(x)+\frac{k}{2}\sigma^2K_1^2}\prod_{i=0}^{k-1} C\left(\frac{\phi^i-1}{\phi-1}\right) = e^{\frac{\phi^k-1}{\phi-1}K_1\psi(x)+A_k+\frac{k}{2}(\sigma^2K_1^2+2B)}\,,$$

$$\sigma^2K_1^2 + 2B = \frac{K_1^2\sigma^2 - 2K_1(\phi-1)[x_0+\sigma^2(1-\gamma)]}{(\phi-1)^2}\,,$$

$$A_k \leqslant A_{\max} = \frac{K_1^2\sigma^2\phi^2}{2(1-\phi)^2(1-\phi^2)} + \frac{K_1(1-\phi)|x_0+\sigma^2(1-\gamma)| + K_1^2\sigma^2|\phi|}{(1-\phi)^3}(1+|\phi|)\,.$$

14. Let $C([-\bar{\nu},\bar{\nu}])$ denote the set of all real-valued continuous functions on the closed interval $[-\bar{\nu},\bar{\nu}]$. For any $f \in C([-\bar{\nu},\bar{\nu}])$, we define $\|f\| := \max_{-\bar{\nu}\leq x\leq\bar{\nu}} |f(x)|$.

15. Equation in step 8 has a unique solution in the space $C([-\bar{\nu},\bar{\nu}])$.

Proof. Let $Q^{(2)}(x)$ and $\bar{Q}^{(2)}(x)$ be solutions of the Equation in step 8 in $C([-\bar{\nu},\bar{\nu}])$. For $-\bar{\nu} \leqslant x \leqslant \bar{\nu}$,

$$Q^{(2)}(x) - \bar{Q}^{(2)}(x) = \frac{K_0 e^{K_1\psi(x)+\frac{1}{2}K_1^2\sigma^2}}{\sqrt{2\pi}\sigma}\int_{-\bar{\nu}}^{\bar{\nu}} e^{-\frac{1}{2\sigma^2}[\nu-\psi(x)-K_1\sigma^2]^2}[Q^{(2)}(\nu) - \bar{Q}^{(2)}(\nu)]\,d\nu.$$

Claim: for all $n = 0, 1, 2, \ldots,$ we have

$$|Q^{(2)}(x) - \bar{Q}^{(2)}(x)| \leqslant \|Q^{(2)} - \bar{Q}^{(2)}\| K_0^n e^{\frac{n}{2}K_1^2\sigma^2} e^{\frac{\phi^n-1}{\phi-1}K_1\psi(x)} \prod_{i=0}^{n-1} C\left(\frac{\phi^i-1}{\phi-1}\right)\,. \tag{$\dagger$}$$

1) If $n = 0$, then we have $|Q^{(2)}(x) - \bar{Q}^{(2)}(x)| \leqslant \|Q^{(2)} - \bar{Q}^{(2)}\|$ for $-\bar{\nu} \leqslant x \leqslant \bar{\nu}$.

2) Suppose that (†) is true for $n \geqslant 0$. By the induction hypothesis, we get

$$
\begin{aligned}
|Q^{(2)}(x)-\bar{Q}^{(2)}(x)| &\leqslant \frac{K_0 e^{K_1\psi(x)+\frac{1}{2}K_1^2\sigma^2}}{\sqrt{2\pi}\sigma}\int_{-\bar{\nu}}^{\bar{\nu}} e^{-\frac{1}{2\sigma^2}[\nu-\psi(x)-K_1\sigma^2]^2}|Q^{(2)}(\nu)-\bar{Q}^{(2)}(\nu)|\,d\nu \\
&\leqslant \|Q^{(2)}-\bar{Q}^{(2)}\| K_0^{n+1} e^{\frac{n+1}{2}K_1^2\sigma^2}\prod_{i=0}^{n-1} C\left(\frac{\phi^i-1}{\phi-1}\right) \\
&\quad \cdot \frac{e^{K_1\psi(x)}}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}[\nu-\psi(x)-K_1\sigma^2]^2} f_{\frac{\phi^n-1}{\phi-1}}(\nu)\,d\nu \\
&\leqslant \|Q^{(2)}-\bar{Q}^{(2)}\| K_0^{n+1} e^{\frac{n+1}{2}K_1^2\sigma^2} e^{K_1\psi(x)} e^{\phi\frac{\phi^n-1}{\phi-1}K_1\psi(x)}\prod_{i=0}^{n} C\left(\frac{\phi^i-1}{\phi-1}\right) \\
&\leqslant \|Q^{(2)}-\bar{Q}^{(2)}\| K_0^{n+1} e^{\frac{n+1}{2}K_1^2\sigma^2} e^{\frac{\phi^{n+1}-1}{\phi-1}K_1\psi(x)}\prod_{i=0}^{n} C\left(\frac{\phi^i-1}{\phi-1}\right).
\end{aligned}
$$

By induction, (†) is true for all $n \in \mathbf{Z}^+$.

$$
\begin{aligned}
|Q^{(2)}(x)-\bar{Q}^{(2)}(x)| &\leqslant \|Q^{(2)}-\bar{Q}^{(2)}\| K_0^n e^{\frac{\phi^n-1}{\phi-1}K_1\psi(x)+A_n+\frac{n}{2}(\sigma^2K_1^2+2B)} \\
&\leqslant \|Q^{(2)}-\bar{Q}^{(2)}\| e^{\frac{\phi^n-1}{\phi-1}K_1\psi(x)+A_{\max}}\left(K_0 e^{\frac{K_1^2\sigma^2-2K_1(\phi-1)[x_0+\sigma^2(1-\gamma)]}{2(\phi-1)^2}}\right)^n.
\end{aligned}
$$

$$
\begin{aligned}
|Q^{(2)}(x)-\bar{Q}^{(2)}(x)| &\leqslant \lim_{n\to\infty}\|Q^{(2)}-\bar{Q}^{(2)}\| e^{\frac{\phi^n-1}{\phi-1}K_1\psi(x)+A_{\max}}\left(K_0 e^{\frac{K_1^2\sigma^2-2K_1(\phi-1)[x_0+\sigma^2(1-\gamma)]}{2(\phi-1)^2}}\right)^n \\
&= \|Q^{(2)}-\bar{Q}^{(2)}\| e^{-\frac{1}{\phi-1}K_1\psi(x)+A_{\max}}\cdot 0 = 0.
\end{aligned}
$$

Therefore, $Q^{(2)}(x)=\bar{Q}^{(2)}(x)$ for $-\bar{\nu} \leqslant x \leqslant \bar{\nu}$. ∎

16. Estimate $Q(x)-Q^{(2)}(x)$ for $-\bar{\nu} \leqslant x \leqslant \bar{\nu}$:

$$
Q(x)-Q^{(2)}(x) = K_0 e^{K_1\psi(x)+\frac{1}{2}K_1^2\sigma^2}\left\{\frac{1}{\sqrt{2\pi}\sigma}\int_{-\bar{\nu}}^{\bar{\nu}} e^{-\frac{1}{2\sigma^2}[\nu-\psi(x)-K_1\sigma^2]^2}[Q(\nu)-Q^{(2)}(\nu)]\,d\nu + E(x)\right\},
$$

where

$$
E(x) = \frac{1}{\sqrt{2\pi}\sigma}\left\{\int_{-\infty}^{-\bar{\nu}} e^{-\frac{1}{2\sigma^2}[\nu-\psi(x)-K_1\sigma^2]^2}Q(\nu)\,d\nu + \int_{\bar{\nu}}^{\infty} e^{-\frac{1}{2\sigma^2}[\nu-\psi(x)-K_1\sigma^2]^2}Q(\nu)\,d\nu\right\}.
$$

17. Since $Q_n(x) \leqslant Q(x)$ for $n \in \mathbf{Z}^+$, we get

$$
\frac{1}{\sqrt{2\pi}\sigma}\left\{\int_{-\infty}^{-\bar{\nu}} e^{-\frac{1}{2\sigma^2}[\nu-\psi(x)-K_1\sigma^2]^2}Q_n(\nu)\,d\nu + \int_{\bar{\nu}}^{\infty} e^{-\frac{1}{2\sigma^2}[\nu-\psi(x)-K_1\sigma^2]^2}Q_n(\nu)\,d\nu\right\} \leqslant E(x) \leqslant \|E\|
$$

for $-\bar{\nu} \leqslant \nu \leqslant \bar{\nu}$.

18. Let x be such that $-\bar\nu \leqslant x \leqslant \bar\nu$. Then

$$Q_{n+1}(x) - Q^{(2)}_{n+1}(x) \leqslant \|E\| \sum_{k=0}^{n} K_0^k e^{\frac{k}{2}K_1^2\sigma^2} e^{\frac{\phi^k-1}{\phi-1}K_1\psi(x)} \prod_{i=0}^{k-1} C\left(\frac{\phi^i-1}{\phi-1}\right) \quad \text{for } n \in \mathbb{Z}^+ . \qquad (\ddagger)$$

Proof. 1) If $n=0$, then $Q(\nu) = Q^{(2)}(\nu) = K_2$ for $-\bar\nu \leqslant \nu \leqslant \bar\nu$ and

$$\begin{aligned}
Q_1(x) - Q_1^{(2)}(x) = {} & \frac{K_0 e^{K_1\psi(x)+\frac{1}{2}K_1^2\sigma^2}}{\sqrt{2\pi}\sigma} \int_{-\bar\nu}^{\bar\nu} e^{-\frac{1}{2\sigma^2}[\nu-\psi(x)-K_1\sigma^2]^2} [Q_0(\nu) - Q_0^{(2)}(\nu)]\, d\nu \\
& + \frac{K_0 e^{K_1\psi(x)+\frac{1}{2}K_1^2\sigma^2}}{\sqrt{2\pi}\sigma} \int_{-\infty}^{-\bar\nu} e^{-\frac{1}{2\sigma^2}[\nu-\psi(x)-K_1\sigma^2]^2} Q_0(\nu)\, d\nu \\
& + \frac{K_0 e^{K_1\psi(x)+\frac{1}{2}K_1^2\sigma^2}}{\sqrt{2\pi}\sigma} \int_{\bar\nu}^{\infty} e^{-\frac{1}{2\sigma^2}[\nu-\psi(x)-K_1\sigma^2]^2} Q_0(\nu)\, d\nu \leqslant E(x) \leqslant \|E\| .
\end{aligned}$$

Since $\|E\| \sum_{k=0}^{0} K_0^k e^{\frac{k}{2}K_1^2\sigma^2} e^{\frac{\phi^k-1}{\phi-1}K_1\psi(x)} \prod_{i=0}^{k-1} C\left(\frac{\phi^i-1}{\phi-1}\right) = \|E\|$, ($\ddagger$) is true.

2) Suppose that ($\ddagger$) is true for $n \geqslant 0$. By the induction hypothesis, we get

$$\begin{aligned}
& Q_{n+2}(x) - Q_{n+2}^{(2)}(x) \\
= {} & \frac{K_0 e^{K_1\psi(x)+\frac{1}{2}K_1^2\sigma^2}}{\sqrt{2\pi}\sigma} \int_{-\bar\nu}^{\bar\nu} e^{-\frac{1}{2\sigma^2}[\nu-\psi(x)-K_1\sigma^2]^2} [Q_{n+1}(\nu) - Q_{n+1}^{(2)}(\nu)]\, d\nu \\
& + \frac{K_0 e^{K_1\psi(x)+\frac{1}{2}K_1^2\sigma^2}}{\sqrt{2\pi}\sigma} \int_{-\infty}^{-\bar\nu} e^{-\frac{1}{2\sigma^2}[\nu-\psi(x)-K_1\sigma^2]^2} Q_{n+1}(\nu)\, d\nu \\
& + \frac{K_0 e^{K_1\psi(x)+\frac{1}{2}K_1^2\sigma^2}}{\sqrt{2\pi}\sigma} \int_{\bar\nu}^{\infty} e^{-\frac{1}{2\sigma^2}[\nu-\psi(x)-K_1\sigma^2]^2} Q_{n+1}(\nu)\, d\nu \\
\leqslant {} & \frac{K_0 e^{K_1\psi(x)+\frac{1}{2}K_1^2\sigma^2}}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}[\nu-\psi(x)-K_1\sigma^2]^2} [Q_{n+1}(\nu) - Q_{n+1}^{(2)}(\nu)]\, d\nu + E(x) \\
\leqslant {} & \|E\| \sum_{k=0}^{n} K_0^{k+1} e^{\frac{k+1}{2}K_1^2\sigma^2} e^{K_1\psi(x)} e^{\phi\frac{\phi^k-1}{\phi-1}K_1\psi(x)} C\left(\frac{\phi^k-1}{\phi-1}\right) \prod_{i=0}^{k-1} C\left(\frac{\phi^i-1}{\phi-1}\right) + \|E\| \\
\leqslant {} & \|E\| \sum_{k=0}^{n} K_0^{k+1} e^{\frac{k+1}{2}K_1^2\sigma^2} e^{\frac{\phi^{k+1}-1}{\phi-1}K_1\psi(x)} \prod_{i=0}^{k} C\left(\frac{\phi^i-1}{\phi-1}\right) + \|E\| \\
= {} & \|E\| \sum_{k=0}^{n+1} K_0^{k} e^{\frac{k}{2}K_1^2\sigma^2} e^{\frac{\phi^{k}-1}{\phi-1}K_1\psi(x)} \prod_{i=0}^{k-1} C\left(\frac{\phi^i-1}{\phi-1}\right) .
\end{aligned}$$

By induction, ($\ddagger$) is true for all $n \in \mathbf{Z}^+$. ∎

19. Estimate $\|Q - Q^{(2)}\|$:

$$Q_{n+1}(x) - Q_{n+1}^{(2)}(x) \leqslant \|E\| e^{\frac{1+|\phi|}{1-\phi}K_1\psi(x)+A_{\max}} \sum_{k=0}^{n} \left(K_0 e^{\frac{K_1^2\sigma^2 - 2K_1(\phi-1)[x_0+\sigma^2(1-\gamma)]}{2(\phi-1)^2}} \right)^k \quad \text{for } -\bar\nu \leqslant x \leqslant \bar\nu ,$$

$$Q(x) - Q^{(2)}(x) \leqslant \|E\| e^{\frac{1+|\phi|}{1-\phi} K_1 \psi(x) + A_{\max}} \sum_{k=0}^{\infty} \left(K_0 e^{\frac{K_1^2 \sigma^2 - 2K_1(\phi-1)[x_0+\sigma^2(1-\gamma)]}{2(\phi-1)^2}} \right)^k ,$$

$$\|Q - Q^{(2)}\| \leqslant \frac{e^{\frac{1+|\phi|}{1-\phi} K_1 [|\phi|\bar{\nu} + x_0 + \sigma^2(1-\gamma)] + A_{\max}}}{1 - K_0 e^{\frac{K_1^2 \sigma^2 - 2K_1(\phi-1)[x_0+\sigma^2(1-\gamma)]}{2(\phi-1)^2}}} \cdot \|E\| .$$

20. Estimate $\|P - P^{(2)}\|$: $P(x) - P^{(2)}(x) = \frac{e^{K_1 x}}{1-K_2 e^{K_1 x}} [Q(x) - Q^{(2)}(x)]$ for $-\bar{\nu} \leqslant x \leqslant \bar{\nu}$.

$$\|P - P^{(2)}\| \leqslant \frac{e^{K_1 \bar{\nu}}}{1 - K_2 e^{K_1 \bar{\nu}}} \|Q - Q^{(2)}\|.$$

∎

Remark. We can show that the integral equation

$$P^{(1)}(x) = \frac{K_0 e^{K_1 x}}{1 - K_2 e^{K_1 x}} M(x) + \frac{1}{\sqrt{2\pi}\sigma} \frac{K_0 e^{K_1 x}}{1 - K_2 e^{K_1 x}} \int_{-\infty}^{\bar{\nu}} e^{-\frac{1}{2\sigma^2}[\nu - \psi(x)]^2} \left(1 - K_2 e^{K_1 \nu}\right) P^{(1)}(\nu)\, d\nu$$

has a unique solution in the space $C((-\infty, \bar{\nu}])$ and $P^{(2)}(x) \leqslant P^{(1)}(x) \leqslant P(x)$ for $-\bar{\nu} \leqslant x \leqslant \bar{\nu}$. Proposition 3 follows by combining steps (19) and (20) with this result.

Figure 1 displays the condition for convergence of the price-dividend function for the Mehra and Prescott case; the price-dividend function exists as long as, $K_4 < 1$. The individual investor's discount rate, $\beta = beta$, is on the x-axis; his (her) coefficient of risk aversion, $\gamma = gam$ is on the y-axis; and $K_4 = K_0 e^{\frac{K_1^2\sigma^2 - 2K_1(\phi-1)[x_0+\sigma^2(1-\gamma)]}{2(\phi-1)^2}}$ is on the z-axis.

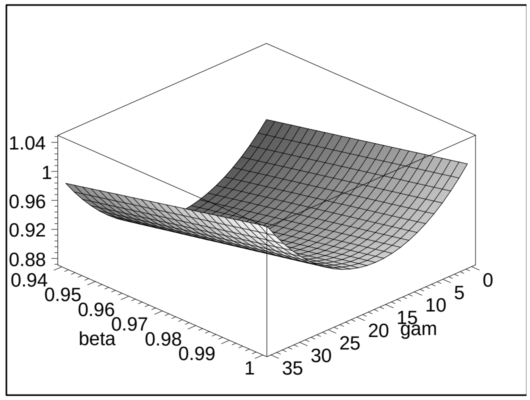

Figure 1

Figure 2 displays three price-dividend functions for the hybrid asset pricing model. The horizontal axis is the current dividend growth rate, x, while the vertical axis is the price-dividend function, $P(x)$ based on Equation (6.15). The flat curve is Mehra-Prescott case, the steepest curve is hybrid habit model with 50% internal habit, and the middle curve is the external habit model. The parameter values used are $\beta = 0.9765$, $\sigma = 0.036$, $\gamma = 3.25$, $x_* = 0.017$, and $\phi = -0.14$.

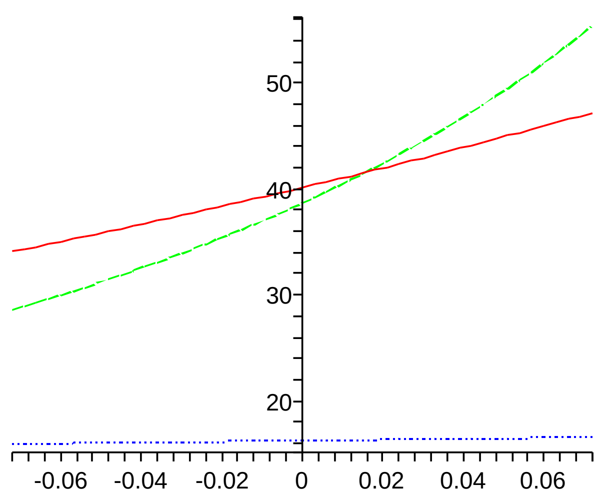

Figure 2

Table 1. Comparison of Results across Cases of the Abel Model

Statistic	Internal 50% $\alpha = 1$ and $\rho = 0.5$	Daily 1962-2003	Monthly 1946-2003	Annual 1871-2002
$E(R)$	0.070	0.055	0.0711	0.070
$\sigma(R)$	0.229	0.151	0.149	0.171
Skewness R	27.280	-19.718	-2.019	-0.584
Kurtosis R	136.553	7835.80	23.920	0.601
$E(R_B)$	0.028	0.016	.011	0.028
$\sigma(R_B)$	0.245	0.002	0.016	0.060
$E(R - R_B)$	0.042	0.039	0.060	0.042
$\sigma(R - R_B)$	0.229	0.151	0.147	0.174
Skewness $R - R_B$	27.280	-19.743	-2.112	-0.544
Kurtosis $R - R_B$	136.553	7860.99	27.167	1.120
$P(.017)$	41.82			22.91

Notes : R is the real return on stocks and R_B is the real return on bonds. C_t is real per capita consumption at time t, and P is the price-dividend ratio. E is the expectation operator and σ is a standard deviation. $P(0.017)$ is the value of the price dividend ratio at the historic average consumption growth rate, 0.017. The statistics for the exact solution are evaluated at the historic average consumption growth rate. The parameter values used are $\beta = 0.9765$, $\sigma = 0.036$, $\gamma = 3.25$, $x_* = 0.017$, and $\phi = -0.14$. We calculate the moments for the distribution of stock (bond) returns by starting with the price-dividend ratio, P, and the price of bonds, converting them to returns, and then integrating to obtain expected returns, standard deviation, skewness and kurtosis. All stock return data refer to the S&P 500 index. The returns are converted to real values by subtracting inflation as measured by the monthly CPI. The daily stock return data are annualized; they are based on CRSP data. The bond return is the 90 day T-Bill rate from FRED II at http://research.stlouisfed.org/. We lost 123 observations for the daily data, because T-Bill rates were missing on these days. The monthly stock return and 90 day T-Bill rate are from CRSP. The annual data from 1871 to 2002 is from Shiller (1989). The updated data was obtained from http://www.econ.yale.edu/∼ shiller/data.htm.

Table 2. Moments of Monthly Nominal Stock Returns

Statistic	$1802 - 2003$	$1802 - 1925$	$1926 - 2003$	$1946 - 2003$
$E(R)$	0.0883	0.0721	0.1162	0.1186
$\sigma(R)$	0.156	0.133	0.186	0.142
Skewness R	0.43	-0.65	0.78	-1.31
Kurtosis R	96.95	46.14	97.71	21.13

Notes : R is the nominal return on stocks. E is the expectation operator and σ is the standard deviation. The sample moments are for Schwert's (1989, 1990) monthly stock return series, found at his website http://schwert.ssb.rocheser.edu/mstock.htm. The data after 1925 is updated following the instructions on Schwert's website, using the CRSP valued weighted market index.

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